http://www.mcs.csueastbay.edu/~kbalasub/reprints/136.pdf
http://www.math.umn.edu/~tlawson/old/18.704/symmetric1.pdf
http://www.polyomino.f2s.com/david/haskell/charactersSn.html
http://en.wikipedia.org/wiki/Symmetric_polynomial
http://www.personal.rhul.ac.uk/usah/080/QITNotes_files/Irreps_v06.pdf
http://mathcircle.berkeley.edu/BMC3/SymPol.pdf
http://faculty.math.tsinghua.edu.cn/~jzhou/SymmetricF.pdf
http://math.stackexchange.com/questions/114151
http:// math.stackexchange.com/questions/395842
http://math.stackexchange.com/questions/83214

special thanks to Marc van Leeuwen for several patient explanations, both by email and on Math StackExchange.

Symmetric Polynomials.

"who can't count doesn't."

■ Old Inits, replaced by 'toolbox' at http://users.telenet.be/Wouter.Meeussen/ToolBox.nb

```
partitionexact[n_, m_] :=
  TransposePartition /@ (Prepend[#1, m] & ) /@ Partitions[n - m, m];
```

```
partitionexact[7, 3]
IntegerPartitions[7, {3}]
```

```
{{3, 2, 2}, {3, 3, 1}, {4, 2, 1}, {5, 1, 1}}
```

```
{{5, 1, 1}, {4, 2, 1}, {3, 3, 1}, {3, 2, 2}}
```

```
TransposePartition /@ Partitions[7, 3]
IntegerPartitions[7, 3]
```

```
{{3, 2, 2}, {3, 3, 1}, {4, 2, 1}, {5, 1, 1}, {4, 3}, {5, 2}, {6, 1}, {7}}
```

```
{{7}, {6, 1}, {5, 2}, {5, 1, 1}, {4, 3}, {4, 2, 1}, {3, 3, 1}, {3, 2, 2}}
```

The test PartitionQ from the *Combinatorica* package does not check if the partition argument is weakly descending, and gives FALSE for partitions with trailing zero(s).
So, the toolbox implements its own version, overwriting the standard one under the same name.

```
PartitionQ /@ {{3, 2, 1}, {3, 1, 2}, {3, 2, 1, 0}}
```

```
{True, True, False}
```

```
?? PartitionQ
```

PartitionQ[p] yields True if p is an integer partition.
PartitionQ[n, p] yields True if p is a partition of n. >>

```
Attributes[PartitionQ] = {Protected}

PartitionQ[Combinatorica`Private`p_List] :=
 Min[Combinatorica`Private`p] > 0 && And @@ IntegerQ /@ Combinatorica`Private`p

PartitionQ[Combinatorica`Private`n_Integer, Combinatorica`Private`p_List] :=
 Plus @@ Combinatorica`Private`p === Combinatorica`Private`n &&
  Min[Combinatorica`Private`p] > 0 && And @@ IntegerQ /@ Combinatorica`Private`p
```

■ Definitions and Implementation conventions.

in the following description we designate the five S.P. by a generic name **u** (representing **m**, **p**, **h**, **e** or **s**).
We choose to define 3 formats for representing the Symmetric Polynomials (S.P.) .
* expanded format: **u[arg** , # of variables **v]** produces Σ a $x_1^i\ x_2^j \dots x_v^z$: allows all symbolic manipulations using standard algebraic functions, but becomes large and slow for small to moderate arguments, and quite unfit for human consumption.
It can however always be reduced to elementary S.P. by the standard '**SymmetricReduction**' function.

* condensed format: $\Sigma_i\, a_i\, p_n{}^i$ where $p_n{}^i$ stands for the i-th partition of n in reverse lexicographic ordering. It codes the expanded format by ignoring the permutation of the (interchangeable) indices and extracting only the (orderless) exponents recast into a partition of n. This format loses the info on the actual number of variables used. It groups monomials according to the exponents : $5\, x_1{}^3\, x_3{}^2$ becomes $5\, p_5{}^3$ since {3,2} is the 3rd partition of 5. The info that there are 3 or more variables 'in play' (as shown by the indices 1 and 3) is lost.

* unevaluated format **uu[arg ,** # of variables **v]** which can be taken as argument for symbolic transformation functions. Only the basic definition **uu[partition, v]==0 /; Length[partition]>v** is coded for (one cannot distribute 5 exponents over 4 variables).

The S.P. **p** , **h** and **e** have the property of being threaded over their (first) argument: **p[{3,1,1}, v]** equals **p[3, v] p[1, v]^2** and the transformation rules work on this last unevaluated form with integer argument. The forward and backward conversions on **uu** are performed by the functions **threadSP[expr , uu]** and **unthreadSP[arg , uu]** regrouping (products of) integer arguments uu[n , $v]^i$ uu[m , $v]^j$ ... into partition arguments uu[{ n (i copies), m (j copies) ... }, v] or the reverse.
The S.P. **m** (monomial) and **s** (Schur) do not share this property, and always need a partition as furst argument. Transformations thus need a threading step in order to get acceptable arguments.

Transformation Functions:
u[arg , # of variables **v]** produces the expanded format Σ a $x_1{}^i\, x_2{}^j \ldots x_v{}^z$,
expr2pow[arg] converts expanded format into condensed format $\Sigma_i\, a_i\, p_n{}^i$,
pow2m[condensed , v :optional **]** converts the condensed format into monomial S.P. but if no **v** (# of variables) is entered, then it defaults to the partition size **n** common to all terms $p_n{}^i$
uu[arg , v] is an inactive (unevaluated) representation except for a partition λ as argument when $|\lambda|>$ **v** (producing 0) ;

u2w[arg , v] converts **uu[arg , v]** into **ww[arg , v]** if such transformation is known and available; it is intended to be implemented as a substitution rule, example: **(expression containing uu[arg , v]) /. uu-> u2w //Expand**

monomProd2Sum[arg] implements the decomposition of powers and products of mm[par, v] into sums of them. (http://math.stackexchange.com/questions/83214)
schurProd2Sum[arg] implements the decomposition of powers and products of ss[par, v] into sums of them by calling the L-R rule package.

The following grid shows the available conversions:
(entries in red need a partition-type argument)

from \ to	e	p	h	m	s
e	1	e2p	e2h	e2m	–
p	p2e	1	p2h	–	p2s
h	h2e	h2p	1	h2m	–
m	m2e	–	–	1	m2s
s	s2e	s2p	s2h	s2m	1

- **Elementary Symmetric Polynomials.**

the function SymmetricReduction can reduce any polynomial in n variables into a sum of a symmetric part in terms of the elementary S.F. and an antisymmetric part.

```
? SymmetricReduction
```

SymmetricReduction[f, {x_1, …, x_n}] gives a pair of polynomials {p, q}
in x_1, …, x_n such that $f == p+q$, where p is the symmetric part and q is the remainder.
SymmetricReduction[f, {x_1, …, x_n}, {s_1, …, s_n}] gives the pair {p, q} with the
elementary symmetric polynomials in p replaced by s_1, …, s_n. ≫

```
First@SymmetricReduction[p[{3, 1, 1}, 4], Array[x# &, 4], Array[ee[#, 4] &, 4]]
```

ee[1, 4]^5 − 3 ee[1, 4]^3 ee[2, 4] + 3 ee[1, 4]^2 ee[3, 4]

```
unthreadSP[% /. ee -> e2p // Expand , pp]
```

pp[{3, 1, 1}, 4]

```
{e[0, 4], e[{}, 4]}
```

{1, 1}

```
Table[e[n, 3], {n, 4}] // ColumnForm
```

$x_1 + x_2 + x_3$
$x_1 x_2 + x_1 x_3 + x_2 x_3$
$x_1 x_2 x_3$
0

```
e[{3, 1, 1}, 4]
% // Expand
```

$(x_1 + x_2 + x_3 + x_4)^2 (x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4)$

$x_1^3 x_2 x_3 + 2 x_1^2 x_2^2 x_3 + x_1 x_2^3 x_3 + 2 x_1^2 x_2 x_3^2 + 2 x_1 x_2^2 x_3^2 + x_1 x_2 x_3^3 + x_1^3 x_2 x_4 +$
$2 x_1^2 x_2^2 x_4 + x_1 x_2^3 x_4 + x_1^3 x_3 x_4 + 7 x_1^2 x_2 x_3 x_4 + 7 x_1 x_2^2 x_3 x_4 + x_2^3 x_3 x_4 + 2 x_1^2 x_3^2 x_4 +$
$7 x_1 x_2 x_3^2 x_4 + 2 x_2^2 x_3^2 x_4 + x_1 x_3^3 x_4 + x_2 x_3^3 x_4 + 2 x_1^2 x_2 x_4^2 + 2 x_1 x_2^2 x_4^2 + 2 x_1^2 x_3 x_4^2 +$
$7 x_1 x_2 x_3 x_4^2 + 2 x_2^2 x_3 x_4^2 + 2 x_1 x_3^2 x_4^2 + 2 x_2 x_3^2 x_4^2 + x_1 x_2 x_4^3 + x_1 x_3 x_4^3 + x_2 x_3 x_4^3$

```
expr2pow[e[{3, 1, 1}, 4]]
```

$12 p_5^4 + 24 p_5^5 + 28 p_5^6$

```
pow2m[%, 4]
```

2 mm[{2, 2, 1}, 4] + mm[{3, 1, 1}, 4] + 7 mm[{2, 1, 1, 1}, 4]

```
threadSP[ ee[{3, 1, 1} , 4], ee]
```

ee[1, 4]^2 ee[3, 4]

```
% /. ee → e2p // Expand
```

$\frac{1}{6}$ pp[1, 4]^5 − $\frac{1}{2}$ pp[1, 4]^3 pp[2, 4] + $\frac{1}{3}$ pp[1, 4]^2 pp[3, 4]

```
unthreadSP[% /. pp → p2e // Expand, ee]
```

ee[{3, 1, 1}, 4]

Power Sum Symmetric Polynomials.

By induction, one finds

$$(13) \qquad k!e_k = \begin{vmatrix} p_1 & p_2 & p_3 & \cdots & p_{k-1} & p_k \\ k-1 & p_1 & p_2 & \cdots & p_{k-2} & p_{k-1} \\ 0 & k-2 & p_1 & \cdots & p_{k-3} & p_{k-2} \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & p_1 & p_2 \\ 0 & 0 & 0 & \cdots & 1 & p_1 \end{vmatrix}$$

One can also rewrite (12) as

$$p_k = \sum_{r=1}^{k-1} (-1)^{k-r-1} e_{k-r} p_r + (-1)^{k-1} k e_k.$$

By induction, one finds

$$(14) \qquad p_k = \begin{vmatrix} e_1 & e_2 & e_3 & \cdots & e_{k-1} & ke_k \\ 1 & e_1 & e_2 & \cdots & e_{k-2} & (k-1)e_{k-1} \\ 0 & 1 & e_1 & \cdots & e_{k-3} & (k-2)e_{k-2} \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & e_1 & 2e_2 \\ 0 & 0 & 0 & \cdots & 1 & e_1 \end{vmatrix}$$

```
p[0, 4]
```

4

```
Table[p[n, 3], {n, 4}] // ColumnForm
```

$x_1 + x_2 + x_3$
$x_1^2 + x_2^2 + x_3^2$
$x_1^3 + x_2^3 + x_3^3$
$x_1^4 + x_2^4 + x_3^4$

```
p[{3, 1, 1}, 4]
% // Expand
```

$(x_1 + x_2 + x_3 + x_4)^2 \left(x_1^3 + x_2^3 + x_3^3 + x_4^3\right)$

$x_1^5 + 2 x_1^4 x_2 + x_1^3 x_2^2 + x_1^2 x_2^3 + 2 x_1 x_2^4 + x_2^5 + 2 x_1^4 x_3 + 2 x_1^3 x_2 x_3 + 2 x_1 x_2^3 x_3 + 2 x_2^4 x_3 + x_1^3 x_3^2 + x_2^3 x_3^2 + x_1^2 x_3^3 + 2 x_1 x_2 x_3^3 + x_2^2 x_3^3 + 2 x_1 x_3^4 + 2 x_2 x_3^4 + x_3^5 + 2 x_1^4 x_4 + 2 x_1^3 x_2 x_4 + 2 x_1 x_2^3 x_4 + 2 x_2^4 x_4 + 2 x_1^3 x_3 x_4 + 2 x_2^3 x_3 x_4 + 2 x_1 x_3^3 x_4 + 2 x_2 x_3^3 x_4 + 2 x_3^4 x_4 + x_1^3 x_4^2 + x_2^3 x_4^2 + x_3^3 x_4^2 + x_1^2 x_4^3 + 2 x_1 x_2 x_4^3 + x_2^2 x_4^3 + 2 x_1 x_3 x_4^3 + 2 x_2 x_3 x_4^3 + x_3^2 x_4^3 + 2 x_1 x_4^4 + 2 x_2 x_4^4 + 2 x_3 x_4^4 + x_4^5$

```
expr2pow[p[{3, 1, 1}, 4]]
```

$4 p_5 + 24 p_5^2 + 12 p_5^3 + 24 p_5^4$

```
{pow2m[%], pow2m[%, 4]}
```

$\left\{\frac{4}{5}\text{mm}[\{5\},5]+\frac{3}{5}\text{mm}[\{3,2\},5]+\frac{6}{5}\text{mm}[\{4,1\},5]+\frac{4}{5}\text{mm}[\{3,1,1\},5],\right.$
$\left.\text{mm}[\{5\},4]+\text{mm}[\{3,2\},4]+2\,\text{mm}[\{4,1\},4]+2\,\text{mm}[\{3,1,1\},4]\right\}$

```
Tr[ pp[#, 4] & /@ {3, 1, 1} ]
```

2 pp[1, 4] + pp[3, 4]

```
% /. pp → p2e // Expand
```

$2\,\text{ee}[1,4]+\text{ee}[1,4]^3-3\,\text{ee}[1,4]\,\text{ee}[2,4]+3\,\text{ee}[3,4]$

```
% /. ee → e2p // Expand
```

2 pp[1, 4] + pp[3, 4]

■ **Complete Homogeneous Symmetric Polynomials.**

INTRODUCTIONS TO SYMMETRIC POLYNOMIALS AND SYMMETRIC FUNCTIONS 15

Inductively one finds:

$$(8)\qquad h_k=\begin{vmatrix} e_1 & e_2 & e_3 & \cdots & e_{k-1} & e_k \\ 1 & e_1 & e_2 & \cdots & e_{k-2} & e_{k-1} \\ 0 & 1 & e_1 & & e_{k-3} & e_{k-2} \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & e_1 & e_2 \\ 0 & 0 & 0 & \cdots & 1 & e_1 \end{vmatrix}=\det(e_{1-i+j})_{1\le i,j\le n}.$$

By symmetry between h and e in the formula (7), one also get

$$(9)\qquad e_k=\begin{vmatrix} h_1 & h_2 & h_3 & \cdots & h_{k-1} & h_k \\ 1 & h_1 & h_2 & \cdots & h_{k-2} & h_{k-1} \\ 0 & 1 & h_1 & & h_{k-3} & h_{k-2} \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & h_1 & h_2 \\ 0 & 0 & 0 & \cdots & 1 & h_1 \end{vmatrix}=\det(h_{1-i+j})_{1\le i,j\le n}.$$

Here we have used the convention that

$$e_i(x_1,\dots,x_n)=0$$

```
Tr[ hh[#, 4] & /@ {3, 1, 1} ]
```

2 hh[1, 4] + hh[3, 4]

```
% /. hh → h2e // Expand
```

$2\,\text{ee}[1,4]+\text{ee}[1,4]^3-2\,\text{ee}[1,4]\,\text{ee}[2,4]+\text{ee}[3,4]$

```
% /. ee → e2h // Expand
```

2 hh[1, 4] + hh[3, 4]

```
Table[h[n, 3], {n, 4}] // ColumnForm
```

$x_1+x_2+x_3$
$x_1^2+x_1x_2+x_2^2+x_1x_3+x_2x_3+x_3^2$
$x_1^3+x_1^2x_2+x_1x_2^2+x_2^3+x_1^2x_3+x_1x_2x_3+x_2^2x_3+x_1x_3^2+x_2x_3^2+x_3^3$
$x_1^4+x_1^3x_2+x_1^2x_2^2+x_1x_2^3+x_2^4+x_1^3x_3+x_1^2x_2x_3+x_1x_2^2x_3+x_2^3x_3+x_1^2x_3^2+x_1x_2x_3^2+x_2^2x_3^2+x_1x_3^3+x_2x_3^3+x_3^4$

```
h[{3, 1, 1}, 4]
% // Expand
```

$$(x_1+x_2+x_3+x_4)^2\left(x_1^3+x_1^2 x_2+x_1 x_2^2+x_2^3+x_1^2 x_3+x_1 x_2 x_3+x_2^2 x_3+x_1 x_3^2+x_2 x_3^2+x_3^3+x_1^2 x_4+x_1 x_2 x_4+x_2^2 x_4+x_1 x_3 x_4+x_2 x_3 x_4+x_3^2 x_4+x_1 x_4^2+x_2 x_4^2+x_3 x_4^2+x_4^3\right)$$

$$x_1^5+3 x_1^4 x_2+4 x_1^3 x_2^2+4 x_1^2 x_2^3+3 x_1 x_2^4+x_2^5+3 x_1^4 x_3+7 x_1^3 x_2 x_3+8 x_1^2 x_2^2 x_3+7 x_1 x_2^3 x_3+3 x_2^4 x_3+4 x_1^3 x_3^2+8 x_1^2 x_2 x_3^2+8 x_1 x_2^2 x_3^2+4 x_2^3 x_3^2+4 x_1^2 x_3^3+7 x_1 x_2 x_3^3+4 x_2^2 x_3^3+3 x_1 x_3^4+3 x_2 x_3^4+x_3^5+3 x_1^4 x_4+7 x_1^3 x_2 x_4+8 x_1^2 x_2^2 x_4+7 x_1 x_2^3 x_4+3 x_2^4 x_4+7 x_1^3 x_3 x_4+13 x_1^2 x_2 x_3 x_4+13 x_1 x_2^2 x_3 x_4+7 x_2^3 x_3 x_4+8 x_1^2 x_3^2 x_4+13 x_1 x_2 x_3^2 x_4+8 x_2^2 x_3^2 x_4+7 x_1 x_3^3 x_4+7 x_2 x_3^3 x_4+3 x_3^4 x_4+4 x_1^3 x_4^2+8 x_1^2 x_2 x_4^2+8 x_1 x_2^2 x_4^2+4 x_2^3 x_4^2+8 x_1^2 x_3 x_4^2+13 x_1 x_2 x_3 x_4^2+8 x_2^2 x_3 x_4^2+8 x_1 x_3^2 x_4^2+8 x_2 x_3^2 x_4^2+4 x_3^3 x_4^2+4 x_1^2 x_4^3+7 x_1 x_2 x_4^3+4 x_2^2 x_4^3+7 x_1 x_3 x_4^3+7 x_2 x_3 x_4^3+4 x_3^2 x_4^3+3 x_1 x_4^4+3 x_2 x_4^4+3 x_3 x_4^4+x_4^5$$

```
expr2pow[h[{3, 1, 1}, 4] ]
```

$$4 p_5+36 p_5^2+48 p_5^3+84 p_5^4+96 p_5^5+52 p_5^6$$

```
pow2m[%, 4]
```

```
mm[{5}, 4] + 4 mm[{3, 2}, 4] + 3 mm[{4, 1}, 4] +
 8 mm[{2, 2, 1}, 4] + 7 mm[{3, 1, 1}, 4] + 13 mm[{2, 1, 1, 1}, 4]
```

an extra relation between monomial and homogenous S.F. :

```
h[3, 4] == Tr[m[#, 4] & /@ Partitions[3]]
```

```
True
```

```
hh[3, 4] hh[1, 4]^2 == Expand[hh[3, 4] hh[1, 4]^2 /. hh → h2m]
```

```
hh[1, 4]^2 hh[3, 4] ==
 mm[{1}, 4]^2 mm[{3}, 4] + mm[{1}, 4]^2 mm[{2, 1}, 4] + mm[{1}, 4]^2 mm[{1, 1, 1}, 4]
```

```
Simplify[% /. hh → h /. mm → m]
```

```
True
```

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and

$$(16)\qquad (-1)^{k-1}p_k=\begin{vmatrix} h_1 & h_2 & h_3 & \cdots & h_{k-1} & kh_k \\ 1 & h_1 & h_2 & \cdots & h_{k-2} & (k-1)h_{k-1} \\ 0 & 1 & h_1 & \cdots & h_{k-3} & (k-2)h_{k-2} \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & h_1 & 2h_2 \\ 0 & 0 & 0 & \cdots & 1 & h_1 \end{vmatrix}.$$

▪ Monomial Symmetric Polynomials.

Takes a partition as argument !

```
{m[{1}, 4], m[{}, 4]}
```

$\{x_1+x_2+x_3+x_4,\ 0\}$

```
m[{1, 1, 1}, 4]
```

$x_1 x_2 x_3+x_1 x_2 x_4+x_1 x_3 x_4+x_2 x_3 x_4$

```
m[#, 4] & /@ Partitions[3]
```

$$\left\{x_1^3+x_2^3+x_3^3+x_4^3,\ x_1^2 x_2+x_1 x_2^2+x_1^2 x_3+x_2^2 x_3+x_1 x_3^2+x_2 x_3^2+x_1^2 x_4+x_2^2 x_4+x_3^2 x_4+x_1 x_4^2+x_2 x_4^2+x_3 x_4^2,\ x_1 x_2 x_3+x_1 x_2 x_4+x_1 x_3 x_4+x_2 x_3 x_4\right\}$$

```
expr2pow[m[{3, 2, 2}, 8]]
```

$168\, p_7^9$

the monomial sym.fun. produces all permutations of $\Pi\ x_i\ \hat{}\ \alpha_i$ and so this condenses to a single exponent-partition,
the coefficient is the multinomial of the partition's content including trailing zero's to make up the variable count.
So, for m[{3,2,2},8] we get 2 two's & 1 three & 5 zero's to make 8 variables, so the coefficient is Multinomial(2,1,5) = 168

```
Multinomial[1, 2, 5]
```

168

without indication of the # of variables v, the default value is the sum of parts (7) and this gives a 8/5 coefficient.
The lost info v=8 needs to be input in order to recover the correct coefficient (=1)

```
{pow2m[%%], pow2m[%%, 8]}
```

$\left\{\frac{8}{5}\,\mathrm{mm}[\{3, 2, 2\}, 7],\ \mathrm{mm}[\{3, 2, 2\}, 8]\right\}$

- **Schur Polynomials.**

Takes a partition as argument !

count of terms:

```
Table[stanley[{n, 2}, n + 2], {n, 2, 9}]
```

{20, 175, 1134, 6468, 34 320, 173 745, 850 850, 4 064 632}

```
λ = {3, 1, 1}; s[λ, 4]
```

$$x_1^3 x_2 x_3 + x_1^2 x_2^2 x_3 + x_1 x_2^3 x_3 + x_1^2 x_2 x_3^2 + x_1 x_2^2 x_3^2 + x_1 x_2 x_3^3 + x_1^3 x_2 x_4 + x_1^2 x_2^2 x_4 + x_1 x_2^3 x_4 + x_1^3 x_3 x_4 + 3 x_1^2 x_2 x_3 x_4 + 3 x_1 x_2^2 x_3 x_4 + x_2^3 x_3 x_4 + x_1^2 x_3^2 x_4 + 3 x_1 x_2 x_3^2 x_4 + x_2^2 x_3^2 x_4 + x_1 x_3^3 x_4 + x_2 x_3^3 x_4 + x_1^2 x_2 x_4^2 + x_1 x_2^2 x_4^2 + x_1^2 x_3 x_4^2 + 3 x_1 x_2 x_3 x_4^2 + x_2^2 x_3 x_4^2 + x_1 x_3^2 x_4^2 + x_2 x_3^2 x_4^2 + x_1 x_2 x_4^3 + x_1 x_3 x_4^3 + x_2 x_3 x_4^3$$

```
expr2pow[%]
```

$12\, p_5^4 + 12\, p_5^5 + 12\, p_5^6$

convert to monomial S.F. via compact form:

```
pow2m[expr2pow[s[λ, 4]], 4]
```

mm[{2, 2, 1}, 4] + mm[{3, 1, 1}, 4] + 3 mm[{2, 1, 1, 1}, 4]

Schur poly's can be defined as $s_\lambda = m_\lambda +$ Sum[$K_{\lambda\mu}\, m_\mu$, $\mu < \lambda$] with $\mu < \lambda$ defined as lesspartitions[λ],
but since the Kostka matrix is 1 on the diagonal and 0 below, this is equivalent to simply $s_\lambda = \Sigma_k K_{\lambda\mu}\, m_\mu$
remark we are dealing with a sum over the partitions $\mu \vdash$ n so n= |λ| = 5, even though we choose to use only 4 variables!

```
lesspartitions[λ]
```

{{2, 2, 1}, {2, 1, 1, 1}, {1, 1, 1, 1, 1}}

```
kostka /@ Partitions[Tr[λ]] // MatrixForm
```

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 & 2 & 3 & 5 \\ 0 & 0 & 0 & 1 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

```
kostka[λ]
```

{0, 0, 0, 1, 1, 3, 6}

```
Expand[ kostka[λ] . (mm[#1, 4] & /@ Partitions[Tr[λ]])] == Expand[
  (mm[#1, 4] & /@ Prepend[lesspartitions[λ], λ]).Drop[kostka[λ], rankpartition[λ] - 1]]
```

True

```
Expand[ kostka[λ] . (m[#1, 4] & /@ Partitions[Tr[λ]])] == s[λ, 4]
```

True

```
Expand[ss[λ, 4] /. ss -> s2h /. hh -> h2p]
```

$\frac{1}{20}\,\mathrm{pp}[1,4]^5-\frac{1}{4}\,\mathrm{pp}[1,4]\,\mathrm{pp}[2,4]^2+\frac{1}{5}\,\mathrm{pp}[5,4]$

```
ss[λ, 4] /. ss → s2p // Expand
```

$\frac{1}{20}\,\mathrm{pp}[1,4]^5-\frac{1}{4}\,\mathrm{pp}[1,4]\,\mathrm{pp}[2,4]^2+\frac{1}{5}\,\mathrm{pp}[5,4]$

```
unthreadSP[%, pp] /. pp → p2s // Expand
```

ss[{3, 1, 1}, 4]

```
% /. ss → s2m // Expand
```

mm[{2, 2, 1}, 4] + mm[{3, 1, 1}, 4] + 3 mm[{2, 1, 1, 1}, 4]

```
s2m[λ, 4]
```

mm[{2, 2, 1}, 4] + mm[{3, 1, 1}, 4] + 3 mm[{2, 1, 1, 1}, 4]

```
% /. mm → m2s // Expand
```

ss[{3, 1, 1}, 4]

- **Timing issues.**

```
Table[Timing[e[{n - 2 , 2}, n];], {n, 4, 16}]
```

{{0., Null}, {0.015625, Null}, {0., Null}, {0., Null}, {0., Null},
{0.015625, Null}, {0.015625, Null}, {0.015625, Null}, {0.031250, Null},
{0.062500, Null}, {0.125000, Null}, {0.250000, Null}, {0.500000, Null}}

```
Table[Timing[p[{n - 2 , 2}, n];], {n, 4, 16}]
```

{{0., Null}, {0., Null}, {0., Null}, {0., Null}, {0., Null}, {0., Null},
{0., Null}, {0., Null}, {0., Null}, {0., Null}, {0., Null}, {0., Null}, {0., Null}}

expanded homogenous S.P. hits the wall at n>=12

```
Table[Timing[h[{n - 2 , 2}, n];], {n, 4, 12}]
```

{{0., Null}, {0., Null}, {0.015625, Null}, {0.031250, Null}, {0.078125, Null},
{0.328125, Null}, {1.343750, Null}, {5.546875, Null}, {23.562500, Null}}

```
Table[Timing[m[{n - 2, 2}, n];], {n, 4, 16}]
```

{{0., Null}, {0., Null}, {0., Null}, {0., Null}, {0., Null},
{0.015625, Null}, {0., Null}, {0.015625, Null}, {0., Null},
{0.015625, Null}, {0., Null}, {0.015625, Null}, {0.015625, Null}}

Schur s[par,v] , Kostka and Character Table char[par] remember previous values:

count of terms in the Schur poly using the hook content formula:

```
Table[stanley[{n - 2, 2}, n], {n, 4, 12}]
```

{20, 175, 1134, 6468, 34 320, 173 745, 850 850, 4 064 632, 19 046 664}

```
Table[Timing[s[{n - 2 , 2}, n];], {n, 4, 7}]
```

{{0.015625, Null}, {0.078125, Null}, {0.640625, Null}, {8.593750, Null}}

the expanded Schur function calculates slowly from n=8 onwards (34320 terms : 210 sec)

```
Timing[s[{8 - 2 , 2}, 8];]
```

```
{210.211347, Null}
```

```
Table[Timing[kostka[{n - 2 , 2}];], {n, 4, 16}]
```

```
{{0.015625, Null}, {0.046875, Null}, {0.062500, Null}, {0.125000, Null}, {0.218750, Null},
 {0.390625, Null}, {0.640625, Null}, {1.000000, Null}, {1.546875, Null},
 {2.343750, Null}, {3.453125, Null}, {5.046875, Null}, {7.515625, Null}}
```

```
Table[Timing[chars[{n - 2 , 2}];], {n, 4, 16}]
```

```
{{0., Null}, {0.015625, Null}, {0.015625, Null}, {0.031250, Null}, {0.046875, Null},
 {0.046875, Null}, {0.078125, Null}, {0.078125, Null}, {0.109375, Null},
 {0.203125, Null}, {0.359375, Null}, {0.703125, Null}, {1.390625, Null}}
```

Since the Kostka numbers calculate more efficiently, it is faster to use **s2m /. mm->m //Expand**

```
Table[Timing[s[{n - 2 , 2}, n] == (s2m[{n - 2, 2}, n] /. mm → m // Expand)], {n, 4, 7}]
```

```
{{0., True}, {0.031200, True}, {0.343202, True}, {6.364841, True}}
```

```
Table[Timing[ (s2m[{n - 2, 2}, n] /. mm → m // Expand);], {n, 4, 7}]
```

```
{{0., Null}, {0.015600, Null}, {0., Null}, {0.031200, Null}}
```

- **Generating Functions.**

$$E(t) := \sum_{i=0}^{n} e_i t^i = \prod_{i=1}^{n} (1 + x_i t),$$

$$H(t) := \sum_{i=0}^{\infty} h_i t^i = \prod_{i=1}^{n} \frac{1}{1 - x_i t},$$

$$P(t) := \sum_{i=1}^{\infty} p_i t^{i-1} = \sum_{i=1}^{n} \frac{x_i}{1 - x_i t}.$$

```
Timing[Coefficient[Product[1 + x_i t, {i, 10}], t^2];]
Timing[e[2, 10];]
```

```
{0., Null}
```

```
{0., Null}
```

remark that the G.F. for the complete Homogeneous symmetric Polynomials H(t) calculates quite inefficiently compared to the combinatorial method using Compositions

```
Timing[
 Coefficient[Series[Product[1 / (1 - Subscript[x, i] t), {i, 10}], {t, 0, 10}], t^4];]
Timing[h[4, 14];]
```

```
{1.812500, Null}
```

```
{0.156250, Null}
```

```
Timing[
 Coefficient[Series[Sum[Subscript[x, i] / (1 - Subscript[x, i] t), {i, 10}], {t, 0, 10}],
    t^4] // Expand ;]
Timing[
 p[
   5,
   10];]
```

```
{0.015625, Null}
```

```
{0., Null}
```

- **Newton's identities, also known as the Newton–Girard formulae, give relations between two types of [symmetric polynomials](), namely between [power sums]() and [elementary symmetric polynomials]().**

generally Sum[(-1)^i p[i, v] e[k - i, v], {i, 0, k}] ==(v-k) e[k, v]
and Sum[-(-1)^i p[i, v] e[k - i, v], {i, 1, k}] == k e[k, v]

```
MatrixForm@Table[Table[ (-1)^i  ee[ k - i, v], {i, 0, 5}] , {k, 0, 5}] ⊗
  MatrixForm@Table[ pp[ i, v] , {i, 0, 5}] ==
 MatrixForm@Table[(v - k) ee[k, v] , {k, 0, 5}]
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ ee[1,v] & -1 & 0 & 0 & 0 & 0 \\ ee[2,v] & -ee[1,v] & 1 & 0 & 0 & 0 \\ ee[3,v] & -ee[2,v] & ee[1,v] & -1 & 0 & 0 \\ ee[4,v] & -ee[3,v] & ee[2,v] & -ee[1,v] & 1 & 0 \\ ee[5,v] & -ee[4,v] & ee[3,v] & -ee[2,v] & ee[1,v] & -1 \end{pmatrix} \otimes \begin{pmatrix} v \\ pp[1,v] \\ pp[2,v] \\ pp[3,v] \\ pp[4,v] \\ pp[5,v] \end{pmatrix} == \begin{pmatrix} v \\ (-1+v)\,ee[1,v] \\ (-2+v)\,ee[2,v] \\ (-3+v)\,ee[3,v] \\ (-4+v)\,ee[4,v] \\ (-5+v)\,ee[5,v] \end{pmatrix}$$

```
% /. MatrixForm → Identity /. CircleTimes → Dot /. pp[0, v] → v /. v → 6 /. pp → p2e //
 Expand
```

```
True
```

```
MatrixForm@Table[Table[ - (-1)^i  ee[ k - i, v], {i, 1, 5}] , {k, 1, 5}] ⊗
  MatrixForm@Table[ pp[ i, v] , {i, 1, 5}] == MatrixForm@Table[(k) ee[k, v] , {k, 1, 5}]
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ ee[1,v] & -1 & 0 & 0 & 0 \\ ee[2,v] & -ee[1,v] & 1 & 0 & 0 \\ ee[3,v] & -ee[2,v] & ee[1,v] & -1 & 0 \\ ee[4,v] & -ee[3,v] & ee[2,v] & -ee[1,v] & 1 \end{pmatrix} \otimes \begin{pmatrix} pp[1,v] \\ pp[2,v] \\ pp[3,v] \\ pp[4,v] \\ pp[5,v] \end{pmatrix} == \begin{pmatrix} ee[1,v] \\ 2\,ee[2,v] \\ 3\,ee[3,v] \\ 4\,ee[4,v] \\ 5\,ee[5,v] \end{pmatrix}$$

```
% /. MatrixForm → Identity /. CircleTimes → Dot /. v → 6 /. pp → p2e // Expand
```

```
True
```

```
Sum[- (-1)^i p[i, 5] e[4 - i, 5], {i, 1, 4}] // Expand
Simplify [% == 4 e[4, 5] ]
```

$4 x_1 x_2 x_3 x_4 + 4 x_1 x_2 x_3 x_5 + 4 x_1 x_2 x_4 x_5 + 4 x_1 x_3 x_4 x_5 + 4 x_2 x_3 x_4 x_5$

```
True
```

```
Sum[(-1)^i pp[i, v] ee[4 - i, v], {i, 0, 4}] // Expand
Expand [% == (v - 4) ee[4, v]  /. ee → e2p  /. v → 5 ]
```

```
v ee[4, v] - ee[3, v] pp[1, v] + ee[2, v] pp[2, v] - ee[1, v] pp[3, v] + pp[4, v]
```

```
True
```

- **The first [Giambelli formula]() gives explicit expression of Schur polynomials as a polynomial in the [complete homogeneous symmetric polynomials]():**
(alias the Jacobi-Trudy identity)

```
μ = {3, 2, 1, 1};
```

```
MatrixForm@Table[u[μ[[i]] - i + j, Length[μ]], {i, Length[μ]}, {j, Length[μ]}]
```

$$\begin{pmatrix} u[3,4] & u[4,4] & u[5,4] & u[6,4] \\ u[1,4] & u[2,4] & u[3,4] & u[4,4] \\ u[-1,4] & u[0,4] & u[1,4] & u[2,4] \\ u[-2,4] & u[-1,4] & u[0,4] & u[1,4] \end{pmatrix}$$

```
MatrixForm@Table[hh[μ[[i]] - i + j, Length[μ]] /. hh[0, _] -> 1 /.
    hh[q_, _] /; q < 0 -> 0, {i, Length[μ]}, {j, Length[μ]}]
```

$$\begin{pmatrix} hh[3,4] & hh[4,4] & hh[5,4] & hh[6,4] \\ hh[1,4] & hh[2,4] & hh[3,4] & hh[4,4] \\ 0 & 1 & hh[1,4] & hh[2,4] \\ 0 & 0 & 1 & hh[1,4] \end{pmatrix}$$

```
Det@Table[hh[μ[[i]] - i + j, Length[μ]] /. hh[0, _] -> 1 /.
    hh[q_, _] /; q < 0 -> 0, {i, Length[μ]}, {j, Length[μ]}]
```

$$hh[1,4]^2\,hh[2,4]\,hh[3,4] - hh[2,4]^2\,hh[3,4] - hh[1,4]\,hh[3,4]^2 - hh[1,4]^3\,hh[4,4] + hh[1,4]\,hh[2,4]\,hh[4,4] + hh[3,4]\,hh[4,4] + hh[1,4]^2\,hh[5,4] - hh[1,4]\,hh[6,4]$$

```
s2h[μ, 4]
```

$$hh[1,4]^2\,hh[2,4]\,hh[3,4] - hh[2,4]^2\,hh[3,4] - hh[1,4]\,hh[3,4]^2 - hh[1,4]^3\,hh[4,4] + hh[1,4]\,hh[2,4]\,hh[4,4] + hh[3,4]\,hh[4,4] + hh[1,4]^2\,hh[5,4] - hh[1,4]\,hh[6,4]$$

```
s[μ, 4] == Expand[% /. hh → h]
```

True

- **The second Giambelli formula gives explicit expression of Schur polynomials as polynomials in the elementary symmetric polynomials.**

```
μ = {3, 1, 0, 0};
ν = PadRight[TransposePartition[μ], Max[Length[TransposePartition[μ]], Length[μ]]]
```

{2, 1, 1, 0}

```
MatrixForm@Table[u[ν[[i]] - i + j, Length[ν]], {i, Length[ν]}, {j, Length[ν]}]
```

$$\begin{pmatrix} u[2,4] & u[3,4] & u[4,4] & u[5,4] \\ u[0,4] & u[1,4] & u[2,4] & u[3,4] \\ u[-1,4] & u[0,4] & u[1,4] & u[2,4] \\ u[-3,4] & u[-2,4] & u[-1,4] & u[0,4] \end{pmatrix}$$

```
MatrixForm@Table[ee[ν[[i]] - i + j, Length[ν]] /. ee[0, _] → 1 /. ee[q_, _] /; q < 0 → 0,
  {i, Length[ν]}, {j, Length[ν]}]
```

$$\begin{pmatrix} ee[2,4] & ee[3,4] & ee[4,4] & ee[5,4] \\ 1 & ee[1,4] & ee[2,4] & ee[3,4] \\ 0 & 1 & ee[1,4] & ee[2,4] \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

```
Expand@Det@Table[e[ν[[i]] - i + j, Length[ν]], {i, Length[ν]}, {j, Length[ν]}]
```

$$x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3 + x_1^3 x_3 + 2 x_1^2 x_2 x_3 + 2 x_1 x_2^2 x_3 + x_2^3 x_3 + x_1^2 x_3^2 + 2 x_1 x_2 x_3^2 + x_2^2 x_3^2 + x_1 x_3^3 + x_2 x_3^3 + x_1^3 x_4 + 2 x_1^2 x_2 x_4 + 2 x_1 x_2^2 x_4 + x_2^3 x_4 + 2 x_1^2 x_3 x_4 + 3 x_1 x_2 x_3 x_4 + 2 x_2^2 x_3 x_4 + 2 x_1 x_3^2 x_4 + 2 x_2 x_3^2 x_4 + x_3^3 x_4 + x_1^2 x_4^2 + 2 x_1 x_2 x_4^2 + x_2^2 x_4^2 + 2 x_1 x_3 x_4^2 + 2 x_2 x_3 x_4^2 + x_3^2 x_4^2 + x_1 x_4^3 + x_2 x_4^3 + x_3 x_4^3$$

```
s[DeleteCases[μ, 0], Max[Length[TransposePartition[μ]], Length[μ]]]
```

$$x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3 + x_1^3 x_3 + 2 x_1^2 x_2 x_3 + 2 x_1 x_2^2 x_3 + x_2^3 x_3 + x_1^2 x_3^2 + 2 x_1 x_2 x_3^2 + x_2^2 x_3^2 + x_1 x_3^3 + x_2 x_3^3 + x_1^3 x_4 + 2 x_1^2 x_2 x_4 + 2 x_1 x_2^2 x_4 + x_2^3 x_4 + 2 x_1^2 x_3 x_4 + 3 x_1 x_2 x_3 x_4 + 2 x_2^2 x_3 x_4 + 2 x_1 x_3^2 x_4 + 2 x_2 x_3^2 x_4 + x_3^3 x_4 + x_1^2 x_4^2 + 2 x_1 x_2 x_4^2 + x_2^2 x_4^2 + 2 x_1 x_3 x_4^2 + 2 x_2 x_3 x_4^2 + x_3^2 x_4^2 + x_1 x_4^3 + x_2 x_4^3 + x_3 x_4^3$$

```
% == %%
```

True

```
s[μ, Max[Length[TransposePartition[μ]], Length[μ]]]
```

$$x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3 + x_1^3 x_3 + 2 x_1^2 x_2 x_3 + 2 x_1 x_2^2 x_3 + x_2^3 x_3 + x_1^2 x_3^2 + 2 x_1 x_2 x_3^2 + x_2^2 x_3^2 + x_1 x_3^3 + x_2 x_3^3 + x_1^3 x_4 + 2 x_1^2 x_2 x_4 + 2 x_1 x_2^2 x_4 + x_2^3 x_4 + 2 x_1^2 x_3 x_4 + 3 x_1 x_2 x_3 x_4 + 2 x_2^2 x_3 x_4 + 2 x_1 x_3^2 x_4 + 2 x_2 x_3^2 x_4 + x_3^3 x_4 + x_1^2 x_4^2 + 2 x_1 x_2 x_4^2 + x_2^2 x_4^2 + 2 x_1 x_3 x_4^2 + 2 x_2 x_3 x_4^2 + x_3^2 x_4^2 + x_1 x_4^3 + x_2 x_4^3 + x_3 x_4^3$$

```
Expand@Det@Table[ee[v[[i]] - i + j, Length[v]], {i, Length[v]}, {j, Length[v]}] /.
   ee[0, _] → 1 /. (ee[q_, _] /; q < 0) → 0 // Simplify
```

$$ee[1, 4]^2 ee[2, 4] - ee[2, 4]^2 - ee[1, 4] ee[3, 4] + ee[4, 4]$$

```
SymmetricReduction[s[DeleteCases[μ, 0], 4], Array[x# &, 4], Array[e# &, 4]]
```

$$\{e_1^2 e_2 - e_2^2 - e_1 e_3 + e_4, 0\}$$

■ Cycleclasses as a more detailed StirlingS1

no nice name yet for function w2[n], use
w2[0] := 1;   w2[n_?Negative] := 0;   w2[n_Integer] := Coefficient[Series[Exp[Sum[s2[k]t^k /k, {k, n}]], {t, 0, n}], t^n] // Expand;

```
(it = Table[w2[k] k ! // Expand , {k, 7}]) // ColumnForm
```

```
s2[1]
s2[1]^2 + s2[2]
s2[1]^3 + 3 s2[1] s2[2] + 2 s2[3]
s2[1]^4 + 6 s2[1]^2 s2[2] + 3 s2[2]^2 + 8 s2[1] s2[3] + 6 s2[4]
s2[1]^5 + 10 s2[1]^3 s2[2] + 15 s2[1] s2[2]^2 + 20 s2[1]^2 s2[3] + 20 s2[2] s2[3] + 30 s2[1] s2[4] + 24 s2
s2[1]^6 + 15 s2[1]^4 s2[2] + 45 s2[1]^2 s2[2]^2 + 15 s2[2]^3 + 40 s2[1]^3 s2[3] + 120 s2[1] s2[2] s2[3] + 4(
s2[1]^7 + 21 s2[1]^5 s2[2] + 105 s2[1]^3 s2[2]^2 + 105 s2[1] s2[2]^3 + 70 s2[1]^4 s2[3] + 420 s2[1]^2 s2[2]
```

```
(Reverse /@ (it /. Plus → List)) /. s2[_] → 1 // ColumnForm
```

```
1
{1, 1}
{2, 3, 1}
{6, 8, 3, 6, 1}
{24, 30, 20, 20, 15, 10, 1}
{120, 144, 90, 90, 40, 120, 40, 15, 45, 15, 1}
{720, 840, 504, 504, 420, 630, 210, 280, 210, 420, 70, 105, 105, 21, 1}
```

```
(Table[cycleclasses[k], {k, 7}]) // ColumnForm
```

```
{1}
{1, 1}
{2, 3, 1}
{6, 8, 3, 6, 1}
{24, 30, 20, 20, 15, 10, 1}
{120, 144, 90, 90, 40, 120, 40, 15, 45, 15, 1}
{720, 840, 504, 504, 420, 630, 210, 280, 210, 420, 70, 105, 105, 21, 1}
```

```
(Table[(-1)^(k + n) StirlingS1[n, k] , {n, 7} , {k, n}]) // ColumnForm
```

```
{1}
{1, 1}
{2, 3, 1}
{6, 11, 6, 1}
{24, 50, 35, 10, 1}
{120, 274, 225, 85, 15, 1}
{720, 1764, 1624, 735, 175, 21, 1}
```

```
Table[Length /@ Partitions[k], {k, 7}] // ColumnForm
```

```
{1}
{1, 2}
{1, 2, 3}
{1, 2, 2, 3, 4}
{1, 2, 2, 3, 3, 4, 5}
{1, 2, 2, 3, 2, 3, 4, 3, 4, 5, 6}
{1, 2, 2, 3, 2, 3, 4, 3, 3, 4, 5, 4, 5, 6, 7}
```

```
(it = Table[Sort@Transpose[{Length /@ Partitions[k], cycleclasses[k]}], {k, 7}]) //
 ColumnForm
```

```
{{1, 1}}
{{1, 1}, {2, 1}}
{{1, 2}, {2, 3}, {3, 1}}
{{1, 6}, {2, 3}, {2, 8}, {3, 6}, {4, 1}}
{{1, 24}, {2, 20}, {2, 30}, {3, 15}, {3, 20}, {4, 10}, {5, 1}}
{{1, 120}, {2, 40}, {2, 90}, {2, 144}, {3, 15}, {3, 90}, {3, 120}, {4, 40}, {4, 45}, {5, 15}, {6
{{1, 720}, {2, 420}, {2, 504}, {2, 840}, {3, 210}, {3, 280}, {3, 504}, {3, 630}, {4, 105}, {4, 2
```

```
(Apply[CirclePlus, Map[Last, Split[#, First[#1] == First[#2] &], {2}], {1}] & /@ it /.
   CirclePlus[i_Integer] → i) // ColumnForm
% /. CirclePlus → Plus
```

```
{1}
{1, 1}
{2, 3, 1}
{6, 3 ⊕ 8, 6, 1}
{24, 20 ⊕ 30, 15 ⊕ 20, 10, 1}
{120, 40 ⊕ 90 ⊕ 144, 15 ⊕ 90 ⊕ 120, 40 ⊕ 45, 15, 1}
{720, 420 ⊕ 504 ⊕ 840, 210 ⊕ 280 ⊕ 504 ⊕ 630, 105 ⊕ 210 ⊕ 420, 70 ⊕ 105, 21, 1}
```

```
{1}
{1, 1}
{2, 3, 1}
{6, 11, 6, 1}
{24, 50, 35, 10, 1}
{120, 274, 225, 85, 15, 1}
{720, 1764, 1624, 735, 175, 21, 1}
```

- **counting SSYT of shape λ with content μ**

http://unapologetic.wordpress.com/2011/02/18/more-kostka-numbers/

allcontents[list] counts the occurrences of 1 ..n in the list

```
allcontents[{{1, 1, 1, 1, 2, 2, 5, 7, 7, 7}}]
```

```
{4, 2, 0, 0, 1, 0, 3}
```

```
λ = {3, 2, 1, 1}; mx = Tr[λ]; rankpartition[λ]
```

```
10
```

```
stanley[λ, mx]
```

```
2940
```

among the SSYT of shape λ, only the last has contents λ

(temp = SSYT[λ, mx]) // tableauxForm

```
 467   466   457   457   456   456   455   455   447   447   447   446   446   446
{57    57    57    56    57    56    57    56    57    56    55    57    56    55
 6   , 6   , 6   , 6   , 6   , 6   , 6   , 6   , 6   , 6   , 6   , 6   , 6   , 6   ,
 7     7     7     7     7     7     7     7     7     7     7     7     7     7

 445   445   445   444   444   444   367   367   367   367   366   366   366   366
 57    56    55    57    56    55    57    47    47    47    57    47    47    47
 6   , 6   , 6   , 6   , 6   , 6   , 6   , 6   , 5   , 5   , 6   , 6   , 5   , 5   ,
 7     7     7     7     7     7     7     7     7     6     7     7     7     6

 357   357   357   357   357   357   357   357   356   356   356   356   356   356
 57    56    47    47    47    46    46    46    57    56    47    47    47    46
 6   , 6   , 6   , 5   , 5   , 6   , 5   , 5   , 6   , 6   , 6   , 5   , 5   , 6   ,
 7     7     7     7     6     7     7     6     7     7     7     7     6     7

 356   356   355   355   355   355   355   355   355   355   347   347
 46    46    57    56    47    47    47    46    46    46    57    56
 5   , 5   , 6   , 6   , 6   , 5   , 5   , 6   , 5   , 5   , 6   , 6   , <<2833>>,
 7     6     7     7     7     7     6     7     7     6     7     7

 111   111   111   111   111   111   111   111   111   111   111   111   111   111
 27    27    27    26    26    26    26    26    26    26    26    26    26    25
 3   , 3   , 3   , 6   , 5   , 5   , 4   , 4   , 4   , 3   , 3   , 3   , 3   , 6   ,
 6     5     4     7     7     6     7     6     5     7     6     5     4     7

 111   111   111   111   111   111   111   111   111   111   111   111   111
 25    25    25    25    25    25    25    25    25    24    24    24    24
 5   , 5   , 4   , 4   , 4   , 3   , 3   , 3   , 3   , 6   , 5   , 5   , 4   ,
 7     6     7     6     5     7     6     5     4     7     7     6     7

 111   111   111   111   111   111   111   111   111   111   111   111   111
 24    24    24    24    24    24    23    23    23    23    23    23    23
 4   , 4   , 3   , 3   , 3   , 3   , 6   , 5   , 5   , 4   , 4   , 4   , 3   ,
 6     5     7     6     5     4     7     7     6     7     6     5     7

 111   111   111   111   111   111   111   111   111   111   111   111   111
 23    23    23    22    22    22    22    22    22    22    22    22    22
 3   , 3   , 3   , 6   , 5   , 5   , 4   , 4   , 4   , 3   , 3   , 3   , 3   }
 6     5     4     7     7     6     7     6     5     7     6     5     4
```

Count[allcontents /@ temp, q_ /; OrderedQ[Reverse[q]]]

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the SSYT of shape λ with content any $\mu \vdash$ n equals the rank(λ) ‘th row of the Kostka matrix

Count[allcontents /@ temp, #] & /@ Partitions[Tr[λ]]

{0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 3, 2, 6, 15, 35}

kostka[λ]

{0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 3, 2, 6, 15, 35}

define ‘zoo’ as the set of all SSYT (with largest element at most mx) of any shape with n = Tr[λ] parts

Tr[stanley[#, mx] & /@ Partitions[Tr@λ]]

43 219

Timing[zoo = SSYT[#, mx] & /@ Partitions[Tr[λ]] ;]

{4.633230, Null}

Length[Flatten[zoo, 1]]

43 219

the count of SSYT with contents μ and any shape $\lambda \vdash$n equals the rankpartition[μ] -th column (rankpartition[λ] th row) in the Kostka matrix:

μ = {2, 2, 2, 1};

```
Map[Count[#, μ] & , Map[allcontents, zoo, {2}] ]
```

```
{1, 3, 5, 3, 4, 6, 1, 3, 3, 2, 0, 1, 0, 0, 0}
```

```
Part[ kostka /@ Partitions[Tr[μ]] , All, rankpartition@μ ]
```

```
{1, 3, 5, 3, 4, 6, 1, 3, 3, 2, 0, 1, 0, 0, 0}
```

```
Map[Cases[#, μ] & , Map[allcontents, zoo, {2}] ]
```

```
{{{2, 2, 2, 1}}, {{2, 2, 2, 1}, {2, 2, 2, 1}, {2, 2, 2, 1}},
 {{2, 2, 2, 1}, {2, 2, 2, 1}, {2, 2, 2, 1}, {2, 2, 2, 1}, {2, 2, 2, 1}},
 {{2, 2, 2, 1}, {2, 2, 2, 1}, {2, 2, 2, 1}},
 {{2, 2, 2, 1}, {2, 2, 2, 1}, {2, 2, 2, 1}, {2, 2, 2, 1}},
 {{2, 2, 2, 1}, {2, 2, 2, 1}, {2, 2, 2, 1}, {2, 2, 2, 1}, {2, 2, 2, 1}, {2, 2, 2, 1}},
 {{2, 2, 2, 1}}, {{2, 2, 2, 1}, {2, 2, 2, 1}, {2, 2, 2, 1}},
 {{2, 2, 2, 1}, {2, 2, 2, 1}, {2, 2, 2, 1}},
 {{2, 2, 2, 1}, {2, 2, 2, 1}}, {}, {{2, 2, 2, 1}}, {}, {}, {}}
```

```
Map[tableauxForm, Select[#, allcontents[#] == μ & ]  & /@ zoo , {1}]
```

```
{{ 1122334 }, { 112334 , 112234 , 112233 },
                2         3         4

 { 11334 , 11234 , 11233 , 11224 , 11223 }, { 11234 , 11233 , 11223 },
   22      23      24      33      34          2       2       3
                                               3       4       4

                                                                              1123
 { 1133 , 1124 , 1123 , 1122 }, { 1134 , 1133 , 1124 , 1123 , 1123 , 1122 }, { 2    },
   224    233    234    334       22     22     23     24     23     33         3
                                  3      4      3      3      4      4          4

                                                 113   112              11
 { 113 , 112 , 112 }, { 114 , 113 , 112 }, { 22  , 23  }, {}, { 22 }, {}, {}, {}}
   224   234   233      22    22    23       3     3            33
   3     3     4        33    34    34       4     4            4
```

- **http://mathoverflow.net/questions/129854 and http://math.stackexchange.com/questions/386936**
- **count the SSYT with sum of entries = n**

```
Table[Tr[Tr[w^Tr[Flatten[#]] & /@ (SSYT[#, 4])] & /@ Partitions[n]], {n, 4}]
Tr[%]
```

$\{w + w^2 + w^3 + w^4,\ w^2 + 2 w^3 + 3 w^4 + 4 w^5 + 3 w^6 + 2 w^7 + w^8,$
$w^3 + 2 w^4 + 4 w^5 + 7 w^6 + 8 w^7 + 8 w^8 + 7 w^9 + 4 w^{10} + 2 w^{11} + w^{12},$
$w^4 + 2 w^5 + 5 w^6 + 9 w^7 + 14 w^8 + 17 w^9 + 20 w^{10} + 17 w^{11} + 14 w^{12} + 9 w^{13} + 5 w^{14} + 2 w^{15} + w^{16}\}$

$w + 2 w^2 + 4 w^3 + 7 w^4 + 10 w^5 + 15 w^6 + 19 w^7 +$
$23 w^8 + 24 w^9 + 24 w^{10} + 19 w^{11} + 15 w^{12} + 9 w^{13} + 5 w^{14} + 2 w^{15} + w^{16}$

```
Tr[w^Tr[Flatten[#]] & /@ (SSYT[#, 4])] & @{3, 1}
```

$w^5 + 2 w^6 + 4 w^7 + 5 w^8 + 7 w^9 + 7 w^{10} + 7 w^{11} + 5 w^{12} + 4 w^{13} + 2 w^{14} + w^{15}$

R. Stanley’s smarter way:

```
s[{3, 1}, 4] /. x_j_ → w^j
```

$w^5 + 2 w^6 + 4 w^7 + 5 w^8 + 7 w^9 + 7 w^{10} + 7 w^{11} + 5 w^{12} + 4 w^{13} + 2 w^{14} + w^{15}$

- **Derivative ∂ s[λ, v] / ∂ p[1, v] = Σ_μ s[μ , v] with μ running over the trimmed λ**

trim[λ_?PartitionQ] := DeleteCases[MapAt[# - 1 &, λ, #], 0] & /@ Position[λ - Append[Rest[λ], 0], _?Positive]

second Giambelli: s --> e

```
s2e[{3, 1}, 4]
```

```
ee[1, 4]^2 ee[2, 4] - ee[2, 4]^2 - ee[1, 4] ee[3, 4] + ee[4, 4]
```

Newton-Girard: e -> p

```
% /. ee → e2p // Expand
```

$\frac{1}{8}$ pp[1, 4]^4 + $\frac{1}{4}$ pp[1, 4]^2 pp[2, 4] − $\frac{1}{8}$ pp[2, 4]^2 − $\frac{1}{4}$ pp[4, 4]

convert s --> e --> p

```
s2p[{3, 1}, 4] // Expand
```

$\frac{1}{8}$ pp[1, 4]^4 + $\frac{1}{4}$ pp[1, 4]^2 pp[2, 4] − $\frac{1}{8}$ pp[2, 4]^2 − $\frac{1}{4}$ pp[4, 4]

```
D[%, pp[1, 4]]
```

$\frac{1}{2}$ pp[1, 4]^3 + $\frac{1}{2}$ pp[1, 4] pp[2, 4]

```
unthreadSP[%, pp] /. pp → p2s // Expand
```

ss[{3}, 4] + ss[{2, 1}, 4]

```
Tr [ ss[#, 4] & /@ trim[{3, 1}] ]
% /. ss → s2p // Expand
```

ss[{3}, 4] + ss[{2, 1}, 4]

$\frac{1}{2}$ pp[1, 4]^3 + $\frac{1}{2}$ pp[1, 4] pp[2, 4]

```
expr2pow[% /. pp → p]
```

$4\, p_3 + 24\, p_3^2 + 12\, p_3^3$

```
pow2m[%, 4]
```

mm[{3}, 4] + 2 mm[{2, 1}, 4] + 3 mm[{1, 1, 1}, 4]

- **check it big time**

```
λ = {5, 4, 3, 2, 1}; v = Tr[λ]; stanley[λ, v]
```

83 622 043 648

the following takes 30 s:

```
s2p[λ, v]
```

$\frac{pp[1,15]^{15}}{4465125} - \frac{pp[1,15]^{12}\,pp[3,15]}{127575} + \frac{pp[1,15]^{9}\,pp[3,15]^2}{25515} - \frac{pp[1,15]^{6}\,pp[3,15]^3}{3645} - \frac{2}{729}pp[1,15]^3\,pp[3,15]^4 + \frac{1}{729}pp[3,15]^5 + \frac{4\,pp[1,15]^{10}\,pp[5,15]}{70875} + \frac{pp[1,15]^{7}\,pp[3,15]\,pp[5,15]}{4725} + \frac{1}{135}pp[1,15]^4\,pp[3,15]^2\,pp[5,15] - \frac{1}{405}pp[1,15]\,pp[3,15]^3\,pp[5,15] - \frac{1}{375}pp[1,15]^5\,pp[5,15]^2 - \frac{1}{75}pp[1,15]^2\,pp[3,15]\,pp[5,15]^2 - \frac{1}{125}pp[5,15]^3 - \frac{pp[1,15]^{8}\,pp[7,15]}{2205} - \frac{1}{315}pp[1,15]^5\,pp[3,15]\,pp[7,15] + \frac{1}{63}pp[1,15]^2\,pp[3,15]^2\,pp[7,15] + \frac{1}{105}pp[1,15]^3\,pp[5,15]\,pp[7,15] + \frac{2}{105}pp[3,15]\,pp[5,15]\,pp[7,15] - \frac{1}{49}pp[1,15]\,pp[7,15]^2 + \frac{1}{405}pp[1,15]^6\,pp[9,15] - \frac{1}{81}pp[1,15]^3\,pp[3,15]\,pp[9,15] - \frac{1}{81}pp[3,15]^2\,pp[9,15] + \frac{1}{45}pp[1,15]\,pp[5,15]\,pp[9,15]$

deriv = D[%, pp[1, v]]

$$\frac{pp[1,15]^{14}}{297675} - \frac{4\,pp[1,15]^{11}\,pp[3,15]}{42525} + \frac{pp[1,15]^{8}\,pp[3,15]^{2}}{2835} - \frac{2\,pp[1,15]^{5}\,pp[3,15]^{3}}{1215} - \frac{2}{243}pp[1,15]^{2}\,pp[3,15]^{4} + \frac{8\,pp[1,15]^{9}\,pp[5,15]}{14175} + \frac{1}{675}pp[1,15]^{6}\,pp[3,15]\,pp[5,15] + \frac{4}{135}pp[1,15]^{3}\,pp[3,15]^{2}\,pp[5,15] - \frac{1}{405}pp[3,15]^{3}\,pp[5,15] - \frac{1}{75}pp[1,15]^{4}\,pp[5,15]^{2} - \frac{2}{75}pp[1,15]\,pp[3,15]\,pp[5,15]^{2} - \frac{8\,pp[1,15]^{7}\,pp[7,15]}{2205} - \frac{1}{63}pp[1,15]^{4}\,pp[3,15]\,pp[7,15] + \frac{2}{63}pp[1,15]\,pp[3,15]^{2}\,pp[7,15] + \frac{1}{35}pp[1,15]^{2}\,pp[5,15]\,pp[7,15] - \frac{1}{49}pp[7,15]^{2} + \frac{2}{135}pp[1,15]^{5}\,pp[9,15] - \frac{1}{27}pp[1,15]^{2}\,pp[3,15]\,pp[9,15] + \frac{1}{45}pp[5,15]\,pp[9,15]$$

unthreadSP[deriv , pp] /. pp → p2s // Expand

```
ss[{5, 4, 3, 2}, 15] + ss[{4, 4, 3, 2, 1}, 15] +
 ss[{5, 3, 3, 2, 1}, 15] + ss[{5, 4, 2, 2, 1}, 15] + ss[{5, 4, 3, 1, 1}, 15]
```

and this does it in less than a second:

Tr [ss[#, v] & /@ trim[λ]]

```
ss[{5, 4, 3, 2}, 15] + ss[{4, 4, 3, 2, 1}, 15] +
 ss[{5, 3, 3, 2, 1}, 15] + ss[{5, 4, 2, 2, 1}, 15] + ss[{5, 4, 3, 1, 1}, 15]
```

deriv == (% /. ss → s2p // Expand)

```
True
```

■ Transformations galore

ee[5, 5];

% /. ee -> e2h

$$hh[1,5]^{5} - 4\,hh[1,5]^{3}\,hh[2,5] + 3\,hh[1,5]\,hh[2,5]^{2} + 3\,hh[1,5]^{2}\,hh[3,5] - 2\,hh[2,5]\,hh[3,5] - 2\,hh[1,5]\,hh[4,5] + hh[5,5]$$

% /. hh → h2e // Expand

```
ee[5, 5]
```

% /. ee -> e2p // Expand

$$\frac{1}{120}pp[1,5]^{5} - \frac{1}{12}pp[1,5]^{3}\,pp[2,5] + \frac{1}{8}pp[1,5]\,pp[2,5]^{2} + \frac{1}{6}pp[1,5]^{2}\,pp[3,5] - \frac{1}{6}pp[2,5]\,pp[3,5] - \frac{1}{4}pp[1,5]\,pp[4,5] + \frac{1}{5}pp[5,5]$$

% /. pp → p2e // Expand

```
ee[5, 5]
```

%% /. pp → p2h // Expand

$$hh[1,5]^{5} - 4\,hh[1,5]^{3}\,hh[2,5] + 3\,hh[1,5]\,hh[2,5]^{2} + 3\,hh[1,5]^{2}\,hh[3,5] - 2\,hh[2,5]\,hh[3,5] - 2\,hh[1,5]\,hh[4,5] + hh[5,5]$$

```
% /. hh → h2p // Expand
```

$$\frac{1}{120}\text{pp}[1,5]^5-\frac{1}{12}\text{pp}[1,5]^3\,\text{pp}[2,5]+\frac{1}{8}\text{pp}[1,5]\,\text{pp}[2,5]^2+\frac{1}{6}\text{pp}[1,5]^2\,\text{pp}[3,5]-\frac{1}{6}\text{pp}[2,5]\,\text{pp}[3,5]-\frac{1}{4}\text{pp}[1,5]\,\text{pp}[4,5]+\frac{1}{5}\text{pp}[5,5]$$

below is a disaster area, but it's correct; don't be tempted to (ab)use 'unthreadSP[__ , mm] on this, because the ss and mm should not be threaded or unthreaded!
we need an extra rule to convert products of monomial S.P. into sums of'm : a kind of Littlewood-Richardson for monomial S.P. :: function **monomProd2Sum**

```
%% /. hh → h2m // Expand
```

```
mm[{1}, 5]^5 - 4 mm[{1}, 5]^3 mm[{2}, 5] + 3 mm[{1}, 5] mm[{2}, 5]^2 +
 3 mm[{1}, 5]^2 mm[{3}, 5] - 2 mm[{2}, 5] mm[{3}, 5] - 2 mm[{1}, 5] mm[{4}, 5] +
 mm[{5}, 5] - 4 mm[{1}, 5]^3 mm[{1, 1}, 5] + 6 mm[{1}, 5] mm[{2}, 5] mm[{1, 1}, 5] -
 2 mm[{3}, 5] mm[{1, 1}, 5] + 3 mm[{1}, 5] mm[{1, 1}, 5]^2 + 3 mm[{1}, 5]^2 mm[{2, 1}, 5] -
 2 mm[{2}, 5] mm[{2, 1}, 5] - 2 mm[{1, 1}, 5] mm[{2, 1}, 5] - 2 mm[{1}, 5] mm[{2, 2}, 5] -
 2 mm[{1}, 5] mm[{3, 1}, 5] + mm[{3, 2}, 5] + mm[{4, 1}, 5] + 3 mm[{1}, 5]^2 mm[{1, 1, 1}, 5] -
 2 mm[{2}, 5] mm[{1, 1, 1}, 5] - 2 mm[{1, 1}, 5] mm[{1, 1, 1}, 5] -
 2 mm[{1}, 5] mm[{2, 1, 1}, 5] + mm[{2, 2, 1}, 5] + mm[{3, 1, 1}, 5] -
 2 mm[{1}, 5] mm[{1, 1, 1, 1}, 5] + mm[{2, 1, 1, 1}, 5] + mm[{1, 1, 1, 1, 1}, 5]
```

```
% /. mm → m2s // Expand
```

```
ss[{1}, 5]^5 - 4 ss[{1}, 5]^3 ss[{2}, 5] + 3 ss[{1}, 5] ss[{2}, 5]^2 +
 3 ss[{1}, 5]^2 ss[{3}, 5] - 2 ss[{2}, 5] ss[{3}, 5] - 2 ss[{1}, 5] ss[{4}, 5] + ss[{5}, 5]
```

```
% /. ss → s2e // Expand
```

```
ee[5, 5]
```

```
%%% // monomProd2Sum
```

```
mm[{1, 1, 1, 1, 1}, 5]
```

```
unthreadSP[ee[5, 5], ee] /. ee -> e2m
```

```
mm[{1, 1, 1, 1, 1}, 5]
```

```
pow2m[expr2pow[e[5, 5]], 5]
```

```
mm[{1, 1, 1, 1, 1}, 5]
```

```
{m[{2, 1}, 3], m[{1}, 3] m[{2}, 3] }
expr2pow /@ %
```

$\{x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_3^2 + x_2 x_3^2,\ (x_1 + x_2 + x_3)\,(x_1^2 + x_2^2 + x_3^2)\}$

$\{6\, p_3^2,\ 3\, p_3 + 6\, p_3^2\}$

```
pow2m /@ %
```

```
{mm[{2, 1}, 3], mm[{3}, 3] + mm[{2, 1}, 3]}
```

clip clip:

For example, there are *integers* $K_{\lambda\mu}$ (called *Kostka numbers*) such that

$$s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu.$$

These numbers are interesting objects to study in algebraic combinatorics. There are also integers χ^λ_μ such that

$$p_\mu = \sum_\lambda \chi^\lambda_\mu s_\lambda, \tag{21}$$

$$s_\lambda = \sum_\mu \frac{\chi^\lambda_\mu}{z_\mu} p_\mu, \tag{22}$$

where

$$z_\mu = \prod_i i^{m_i(\mu)} m_i(\mu)!.$$

The integers $\{\chi^\lambda_\mu\}$ give the character table of the symmetric groups. For details, see §??.

so we can get a 'p2s' too: it only works on an entire partition argument

we need the dot product between ss[all-λ] and the rank-μ th column of the character table of Sn

```
Apply[Times, pp[#, 5] & /@ {3, 2}] ==
 (Part[chars[#], rankpartition[{3, 2}]] & /@ Partitions[5]) . (ss[#, 5] & /@ Partitions[5])
```

```
pp[2, 5] pp[3, 5] == ss[{5}, 5] + ss[{3, 2}, 5] - ss[{4, 1}, 5] -
  ss[{2, 2, 1}, 5] + ss[{2, 1, 1, 1}, 5] - ss[{1, 1, 1, 1, 1}, 5]
```

```
% /. pp → p /. ss → s  // Expand
```

```
True
```

```
%% /. pp → p2s // Expand
```

```
p2s[2, 5] p2s[3, 5] == ss[{5}, 5] + ss[{3, 2}, 5] -
  ss[{4, 1}, 5] - ss[{2, 2, 1}, 5] + ss[{2, 1, 1, 1}, 5] - ss[{1, 1, 1, 1, 1}, 5]
```

```
%%% /. ss → s2p // Expand
```

```
True
```

```
p2s[{3, 2}, 5]
```

```
ss[{5}, 5] + ss[{3, 2}, 5] - ss[{4, 1}, 5] -
 ss[{2, 2, 1}, 5] + ss[{2, 1, 1, 1}, 5] - ss[{1, 1, 1, 1, 1}, 5]
```

```
ss[#, 5] & /@ Partitions[5] /. ss -> s2e /. ee → e2p // Expand
```

{$\frac{1}{120}$ pp[1, 5]^5 + $\frac{1}{12}$ pp[1, 5]^3 pp[2, 5] + $\frac{1}{8}$ pp[1, 5] pp[2, 5]^2 +
$\frac{1}{6}$ pp[1, 5]^2 pp[3, 5] + $\frac{1}{6}$ pp[2, 5] pp[3, 5] + $\frac{1}{4}$ pp[1, 5] pp[4, 5] + $\frac{1}{5}$ pp[5, 5],
$\frac{1}{30}$ pp[1, 5]^5 + $\frac{1}{6}$ pp[1, 5]^3 pp[2, 5] + $\frac{1}{6}$ pp[1, 5]^2 pp[3, 5] − $\frac{1}{6}$ pp[2, 5] pp[3, 5] − $\frac{1}{5}$ pp[5, 5],
$\frac{1}{24}$ pp[1, 5]^5 + $\frac{1}{12}$ pp[1, 5]^3 pp[2, 5] + $\frac{1}{8}$ pp[1, 5] pp[2, 5]^2 −
$\frac{1}{6}$ pp[1, 5]^2 pp[3, 5] + $\frac{1}{6}$ pp[2, 5] pp[3, 5] − $\frac{1}{4}$ pp[1, 5] pp[4, 5],
$\frac{1}{20}$ pp[1, 5]^5 − $\frac{1}{4}$ pp[1, 5] pp[2, 5]^2 + $\frac{1}{5}$ pp[5, 5], $\frac{1}{24}$ pp[1, 5]^5 − $\frac{1}{12}$ pp[1, 5]^3 pp[2, 5] +
$\frac{1}{8}$ pp[1, 5] pp[2, 5]^2 − $\frac{1}{6}$ pp[1, 5]^2 pp[3, 5] − $\frac{1}{6}$ pp[2, 5] pp[3, 5] + $\frac{1}{4}$ pp[1, 5] pp[4, 5],
$\frac{1}{30}$ pp[1, 5]^5 − $\frac{1}{6}$ pp[1, 5]^3 pp[2, 5] + $\frac{1}{6}$ pp[1, 5]^2 pp[3, 5] + $\frac{1}{6}$ pp[2, 5] pp[3, 5] − $\frac{1}{5}$ pp[5, 5],
$\frac{1}{120}$ pp[1, 5]^5 − $\frac{1}{12}$ pp[1, 5]^3 pp[2, 5] + $\frac{1}{8}$ pp[1, 5] pp[2, 5]^2 +
$\frac{1}{6}$ pp[1, 5]^2 pp[3, 5] − $\frac{1}{6}$ pp[2, 5] pp[3, 5] − $\frac{1}{4}$ pp[1, 5] pp[4, 5] + $\frac{1}{5}$ pp[5, 5]}

```
unthreadSP[#, pp] & /@ % // ColumnForm
```

$\frac{1}{5}$ pp[{5}, 5] + $\frac{1}{6}$ pp[{3, 2}, 5] + $\frac{1}{4}$ pp[{4, 1}, 5] + $\frac{1}{8}$ pp[{2, 2, 1}, 5] + $\frac{1}{6}$ pp[{3, 1, 1}, 5] + $\frac{1}{12}$ pp[{
−$\frac{1}{5}$ pp[{5}, 5] − $\frac{1}{6}$ pp[{3, 2}, 5] + $\frac{1}{6}$ pp[{3, 1, 1}, 5] + $\frac{1}{6}$ pp[{2, 1, 1, 1}, 5] + $\frac{1}{30}$ pp[{1, 1, 1, 1, 1
$\frac{1}{6}$ pp[{3, 2}, 5] − $\frac{1}{4}$ pp[{4, 1}, 5] + $\frac{1}{8}$ pp[{2, 2, 1}, 5] − $\frac{1}{6}$ pp[{3, 1, 1}, 5] + $\frac{1}{12}$ pp[{2, 1, 1, 1}, 5]
$\frac{1}{5}$ pp[{5}, 5] − $\frac{1}{4}$ pp[{2, 2, 1}, 5] + $\frac{1}{20}$ pp[{1, 1, 1, 1, 1}, 5]
−$\frac{1}{6}$ pp[{3, 2}, 5] + $\frac{1}{4}$ pp[{4, 1}, 5] + $\frac{1}{8}$ pp[{2, 2, 1}, 5] − $\frac{1}{6}$ pp[{3, 1, 1}, 5] − $\frac{1}{12}$ pp[{2, 1, 1, 1},
−$\frac{1}{5}$ pp[{5}, 5] + $\frac{1}{6}$ pp[{3, 2}, 5] + $\frac{1}{6}$ pp[{3, 1, 1}, 5] − $\frac{1}{6}$ pp[{2, 1, 1, 1}, 5] + $\frac{1}{30}$ pp[{1, 1, 1, 1, 1
$\frac{1}{5}$ pp[{5}, 5] − $\frac{1}{6}$ pp[{3, 2}, 5] − $\frac{1}{4}$ pp[{4, 1}, 5] + $\frac{1}{8}$ pp[{2, 2, 1}, 5] + $\frac{1}{6}$ pp[{3, 1, 1}, 5] − $\frac{1}{12}$ pp[{

```
Transpose[Inverse[chars /@ Partitions[5]]] // Grid
```

$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{120}$
$-\frac{1}{5}$	0	$-\frac{1}{6}$	$\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{30}$
0	$-\frac{1}{4}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{24}$
$\frac{1}{5}$	0	0	0	$-\frac{1}{4}$	0	$\frac{1}{20}$
0	$\frac{1}{4}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{8}$	$-\frac{1}{12}$	$\frac{1}{24}$
$-\frac{1}{5}$	0	$\frac{1}{6}$	$\frac{1}{6}$	0	$-\frac{1}{6}$	$\frac{1}{30}$
$\frac{1}{5}$	$-\frac{1}{4}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}$	$-\frac{1}{12}$	$\frac{1}{120}$

```
Expand[ss[#, 5] & /@ Partitions[5] /. ss -> s2e /. ee → e2p ] ==
 Expand[ss[#, 5] & /@ Partitions[5] /. ss -> s2p ]
```

True

- **http://math.stackexchange.com/questions/83214**

Elementary

Is there a General Formula for the Transition Matrix from Products of Symmetric Polynomials to Monomial Symmetric Functions?

7

1

Given the elementary symmetric polynomials $e_k(X_1, X_2, \ldots, X_N)$ generated via

$$\prod_{k=1}^{N}(t+X_k) = e_0 t^N + e_1 t^{N-1} + \cdots + e_N.$$

How can one get the monomial symmetric functions $m_\lambda(X_1, X_2, \ldots, X_N)$ as products and sums in e_k? For example: $N = 4$

$$m_{(2,1,1,0)} = X_1^2 X_2 X_3 + \text{all permutations} = e_3 \cdot e_1 - 4e_4,$$

$$m_{(2,2,0,0)} = X_1^2 X_2^2 + \ldots = e_2^2 - 6e_4 - 2m_{(2,1,1,0)} = e_2^2 - 2e_3 \cdot e_1 + 2e_4$$

It seems clear that the products on the RHS run over all partitions μ of N. For each λ there should be a set of integers $C_{\lambda\mu}$ such that

$$m_\lambda = \sum_\mu C_{\lambda\mu} \prod_j e_{\mu_j}$$

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$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 2 & 6 \\ 0 & 1 & 2 & 5 & 12 \\ 1 & 4 & 6 & 12 & 24 \end{pmatrix} \begin{pmatrix} m_{4,0,0,0} \\ m_{3,1,0,0} \\ m_{2,2,0,0} \\ m_{2,1,1,0} \\ m_{1,1,1,1} \end{pmatrix} = \begin{pmatrix} e_4 \\ e_3 e_1 \\ e_2^2 \\ e_2 e_1^2 \\ e_1^4 \end{pmatrix}.$$

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2 Answers   active oldest **votes**

7

+50

Probably a simple general formula for your matrix $C_n = (C_{\lambda,\mu})_{\lambda,\mu \in \mathcal{P}_n}$, where $\mathcal{P}_n$ denotes the partitions of n (ordered in decreasing lexicographic ordering) does not exist. However a number of things can be said, notably your above guesses can be confirmed. One thing that your examples suggest but which is false is that the matrix is "upper-left triangular", which fails from $n = 6$ on; the reason is that the true triangularity is given by $C_{\lambda,\mu} \neq 0 \implies \lambda^{tr} \leq \mu$ in the dominance order where λ^{tr} is the transpose (conjugate) partition of λ, and lexicographic order for $n \geq 6$ does not assign complementary positions to λ and λ^{tr} (nor does *any* ordering for n sufficiently large).

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One could either compute $M_{\lambda,\mu}$ this way and invert the result, or invert the matrix of Kostka numbers and deduce from above formula, which can be interpreted as a matrix product, the formula

$$C_{\lambda,\mu} = \sum_{\lambda^{tr} \leq \nu \leq \mu} K'_{\lambda,\nu^{tr}} K'_{\mu,\nu}$$

where $(K'_{\lambda,\mu})_{\lambda,\mu \in \mathcal{P}_n}$ is the inverse of the Kostka matrix $(K_{\lambda,\mu})_{\lambda,\mu \in \mathcal{P}_n}$. I don't know very much about these "inverse Kostka numbers", but you will find some information about them in the answers to this MO question; I'm not sure this allows them to be computed more efficiently than by inverting the Kostka matrix.

share improve this answer   edited Jan 7 '12 at 13:53   answered Jan 7 '12 at 13:15
Marc van Leeuwen
23.7k ● 1 ■ 13 ▲ 53

```
MatrixForm[ ko = kostka /@ Partitions[6] ]
```

1	1	1	1	1	1	1	1	1	1	1
0	1	1	2	1	2	3	2	3	4	5
0	0	1	1	1	2	3	3	4	6	9
0	0	0	1	0	1	3	1	3	6	10
0	0	0	0	1	1	1	1	2	3	5
0	0	0	0	0	1	2	2	4	8	16
0	0	0	0	0	0	1	0	1	4	10
0	0	0	0	0	0	0	1	1	2	5
0	0	0	0	0	0	0	0	1	3	9
0	0	0	0	0	0	0	0	0	1	5
0	0	0	0	0	0	0	0	0	0	1

```
MatrixForm[ ok = kostka[TransposePartition[#]] & /@ Partitions[6] ]
```

0	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	1	5
0	0	0	0	0	0	0	0	1	3	9
0	0	0	0	0	0	1	0	1	4	10
0	0	0	0	0	0	0	1	1	2	5
0	0	0	0	0	1	2	2	4	8	16
0	0	0	1	0	1	3	1	3	6	10
0	0	0	0	1	1	1	1	2	3	5
0	0	1	1	1	2	3	3	4	6	9
0	1	1	2	1	2	3	2	3	4	5
1	1	1	1	1	1	1	1	1	1	1

```
MatrixForm[ Transpose[ok].ko ]
```

0	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	1	6
0	0	0	0	0	0	0	0	1	4	15
0	0	0	0	0	0	1	0	2	9	30
0	0	0	0	0	0	0	1	2	6	20
0	0	0	0	0	1	3	3	8	22	60
0	0	0	1	0	3	10	6	18	48	120
0	0	0	0	1	3	6	6	15	36	90
0	0	1	2	2	8	18	15	34	78	180
0	1	4	9	6	22	48	36	78	168	360
1	6	15	30	20	60	120	90	180	360	720

Lawson's reducible representations:

```
MatrixForm[ Transpose[Inverse[(cycleclasses[6] #) & /@ (chars /@ Partitions[6])].
   (6 ! kostka /@ Partitions[6])] ]
```

1	1	1	1	1	1	1	1	1	1	1
0	1	0	2	0	1	3	0	2	4	6
0	0	1	1	0	1	3	3	3	7	15
0	0	0	2	0	0	6	0	2	12	30
0	0	0	0	2	2	2	0	4	8	20
0	0	0	0	0	1	3	0	4	16	60
0	0	0	0	0	0	6	0	0	24	120
0	0	0	0	0	0	0	6	6	18	90
0	0	0	0	0	0	0	0	4	24	180
0	0	0	0	0	0	0	0	0	24	360
0	0	0	0	0	0	0	0	0	0	720

```
threadSP[ee[#, 6], ee] & /@ Partitions[6]
```

```
{ee[6, 6], ee[1, 6] ee[5, 6], ee[2, 6] ee[4, 6], ee[1, 6]^2 ee[4, 6],
 ee[3, 6]^2, ee[1, 6] ee[2, 6] ee[3, 6], ee[1, 6]^3 ee[3, 6],
 ee[2, 6]^3, ee[1, 6]^2 ee[2, 6]^2, ee[1, 6]^4 ee[2, 6], ee[1, 6]^6}
```

```
% /. ee → e2h // Expand
```

```
{hh[1, 6]^6 - 5 hh[1, 6]^4 hh[2, 6] + 6 hh[1, 6]^2 hh[2, 6]^2 -
  hh[2, 6]^3 + 4 hh[1, 6]^3 hh[3, 6] - 6 hh[1, 6] hh[2, 6] hh[3, 6] + hh[3, 6]^2 -
  3 hh[1, 6]^2 hh[4, 6] + 2 hh[2, 6] hh[4, 6] + 2 hh[1, 6] hh[5, 6] - hh[6, 6],
 hh[1, 6]^6 - 4 hh[1, 6]^4 hh[2, 6] + 3 hh[1, 6]^2 hh[2, 6]^2 + 3 hh[1, 6]^3 hh[3, 6] -
  2 hh[1, 6] hh[2, 6] hh[3, 6] - 2 hh[1, 6]^2 hh[4, 6] + hh[1, 6] hh[5, 6],
 hh[1, 6]^6 - 4 hh[1, 6]^4 hh[2, 6] + 4 hh[1, 6]^2 hh[2, 6]^2 - hh[2, 6]^3 + 2 hh[1, 6]^3 hh[3, 6] -
  2 hh[1, 6] hh[2, 6] hh[3, 6] - hh[1, 6]^2 hh[4, 6] + hh[2, 6] hh[4, 6],
 hh[1, 6]^6 - 3 hh[1, 6]^4 hh[2, 6] + hh[1, 6]^2 hh[2, 6]^2 + 2 hh[1, 6]^3 hh[3, 6] -
  hh[1, 6]^2 hh[4, 6], hh[1, 6]^6 - 4 hh[1, 6]^4 hh[2, 6] + 4 hh[1, 6]^2 hh[2, 6]^2 +
  2 hh[1, 6]^3 hh[3, 6] - 4 hh[1, 6] hh[2, 6] hh[3, 6] + hh[3, 6]^2,
 hh[1, 6]^6 - 3 hh[1, 6]^4 hh[2, 6] + 2 hh[1, 6]^2 hh[2, 6]^2 + hh[1, 6]^3 hh[3, 6] -
  hh[1, 6] hh[2, 6] hh[3, 6], hh[1, 6]^6 - 2 hh[1, 6]^4 hh[2, 6] + hh[1, 6]^3 hh[3, 6],
 hh[1, 6]^6 - 3 hh[1, 6]^4 hh[2, 6] + 3 hh[1, 6]^2 hh[2, 6]^2 - hh[2, 6]^3,
 hh[1, 6]^6 - 2 hh[1, 6]^4 hh[2, 6] + hh[1, 6]^2 hh[2, 6]^2,
 hh[1, 6]^6 - hh[1, 6]^4 hh[2, 6], hh[1, 6]^6}
```

```
% /. hh → h2m // Expand
```

```
{mm[{1}, 6]^6 - 5 mm[{1}, 6]^4 mm[{2}, 6] + 6 mm[{1}, 6]^2 mm[{2}, 6]^2 - mm[{2}, 6]^3 +
  4 mm[{1}, 6]^3 mm[{3}, 6] - 6 mm[{1}, 6] mm[{2}, 6] mm[{3}, 6] + mm[{3}, 6]^2 -
  3 mm[{1}, 6]^2 mm[{4}, 6] + 2 mm[{2}, 6] mm[{4}, 6] + 2 mm[{1}, 6] mm[{5}, 6] -
  mm[{6}, 6] - 5 mm[{1}, 6]^4 mm[{1, 1}, 6] + 12 mm[{1}, 6]^2 mm[{2}, 6] mm[{1, 1}, 6] -
  3 mm[{2}, 6]^2 mm[{1, 1}, 6] - 6 mm[{1}, 6] mm[{3}, 6] mm[{1, 1}, 6] +
  2 mm[{4}, 6] mm[{1, 1}, 6] + 6 mm[{1}, 6]^2 mm[{1, 1}, 6]^2 - 3 mm[{2}, 6] mm[{1, 1}, 6]^2 -
  mm[{1, 1}, 6]^3 + 4 mm[{1}, 6]^3 mm[{2, 1}, 6] - 6 mm[{1}, 6] mm[{2}, 6] mm[{2, 1}, 6] +
  2 mm[{3}, 6] mm[{2, 1}, 6] - 6 mm[{1}, 6] mm[{1, 1}, 6] mm[{2, 1}, 6] + mm[{2, 1}, 6]^2 -
  3 mm[{1}, 6]^2 mm[{2, 2}, 6] + 2 mm[{2}, 6] mm[{2, 2}, 6] + 2 mm[{1, 1}, 6] mm[{2, 2}, 6] -
  3 mm[{1}, 6]^2 mm[{3, 1}, 6] + 2 mm[{2}, 6] mm[{3, 1}, 6] + 2 mm[{1, 1}, 6] mm[{3, 1}, 6] +
  2 mm[{1}, 6] mm[{3, 2}, 6] - mm[{3, 3}, 6] + 2 mm[{1}, 6] mm[{4, 1}, 6] - mm[{4, 2}, 6] -
  mm[{5, 1}, 6] + 4 mm[{1}, 6]^3 mm[{1, 1, 1}, 6] - 6 mm[{1}, 6] mm[{2}, 6] mm[{1, 1, 1}, 6] +
  2 mm[{3}, 6] mm[{1, 1, 1}, 6] - 6 mm[{1}, 6] mm[{1, 1}, 6] mm[{1, 1, 1}, 6] +
  2 mm[{2, 1}, 6] mm[{1, 1, 1}, 6] + mm[{1, 1, 1}, 6]^2 - 3 mm[{1}, 6]^2 mm[{2, 1, 1}, 6] +
  2 mm[{2}, 6] mm[{2, 1, 1}, 6] + 2 mm[{1, 1}, 6] mm[{2, 1, 1}, 6] +
  2 mm[{1}, 6] mm[{2, 2, 1}, 6] - mm[{2, 2, 2}, 6] + 2 mm[{1}, 6] mm[{3, 1, 1}, 6] -
  mm[{3, 2, 1}, 6] - mm[{4, 1, 1}, 6] - 3 mm[{1}, 6]^2 mm[{1, 1, 1, 1}, 6] +
  2 mm[{2}, 6] mm[{1, 1, 1, 1}, 6] + 2 mm[{1, 1}, 6] mm[{1, 1, 1, 1}, 6] +
  2 mm[{1}, 6] mm[{2, 1, 1, 1}, 6] - mm[{2, 2, 1, 1}, 6] - mm[{3, 1, 1, 1}, 6] +
  2 mm[{1}, 6] mm[{1, 1, 1, 1, 1}, 6] - mm[{2, 1, 1, 1, 1}, 6] - mm[{1, 1, 1, 1, 1, 1}, 6],
 mm[{1}, 6]^6 - 4 mm[{1}, 6]^4 mm[{2}, 6] + 3 mm[{1}, 6]^2 mm[{2}, 6]^2 + 3 mm[{1}, 6]^3 mm[{3}, 6] -
  2 mm[{1}, 6] mm[{2}, 6] mm[{3}, 6] - 2 mm[{1}, 6]^2 mm[{4}, 6] + mm[{1}, 6] mm[{5}, 6] -
  4 mm[{1}, 6]^4 mm[{1, 1}, 6] + 6 mm[{1}, 6]^2 mm[{2}, 6] mm[{1, 1}, 6] -
  2 mm[{1}, 6] mm[{3}, 6] mm[{1, 1}, 6] + 3 mm[{1}, 6]^2 mm[{1, 1}, 6]^2 +
  3 mm[{1}, 6]^3 mm[{2, 1}, 6] - 2 mm[{1}, 6] mm[{2}, 6] mm[{2, 1}, 6] -
  2 mm[{1}, 6] mm[{1, 1}, 6] mm[{2, 1}, 6] - 2 mm[{1}, 6]^2 mm[{2, 2}, 6] -
  2 mm[{1}, 6]^2 mm[{3, 1}, 6] + mm[{1}, 6] mm[{3, 2}, 6] + mm[{1}, 6] mm[{4, 1}, 6] +
  3 mm[{1}, 6]^3 mm[{1, 1, 1}, 6] - 2 mm[{1}, 6] mm[{2}, 6] mm[{1, 1, 1}, 6] -
  2 mm[{1}, 6] mm[{1, 1}, 6] mm[{1, 1, 1}, 6] - 2 mm[{1}, 6]^2 mm[{2, 1, 1}, 6] +
  mm[{1}, 6] mm[{2, 2, 1}, 6] + mm[{1}, 6] mm[{3, 1, 1}, 6] - 2 mm[{1}, 6]^2 mm[{1, 1, 1, 1}, 6] +
  mm[{1}, 6] mm[{2, 1, 1, 1}, 6] + mm[{1}, 6] mm[{1, 1, 1, 1, 1}, 6],
 mm[{1}, 6]^6 - 4 mm[{1}, 6]^4 mm[{2}, 6] + 4 mm[{1}, 6]^2 mm[{2}, 6]^2 - mm[{2}, 6]^3 +
  2 mm[{1}, 6]^3 mm[{3}, 6] - 2 mm[{1}, 6] mm[{2}, 6] mm[{3}, 6] - mm[{1}, 6]^2 mm[{4}, 6] +
  mm[{2}, 6] mm[{4}, 6] - 4 mm[{1}, 6]^4 mm[{1, 1}, 6] + 8 mm[{1}, 6]^2 mm[{2}, 6] mm[{1, 1}, 6] -
  3 mm[{2}, 6]^2 mm[{1, 1}, 6] - 2 mm[{1}, 6] mm[{3}, 6] mm[{1, 1}, 6] +
  mm[{4}, 6] mm[{1, 1}, 6] + 4 mm[{1}, 6]^2 mm[{1, 1}, 6]^2 - 3 mm[{2}, 6] mm[{1, 1}, 6]^2 -
  mm[{1, 1}, 6]^3 + 2 mm[{1}, 6]^3 mm[{2, 1}, 6] - 2 mm[{1}, 6] mm[{2}, 6] mm[{2, 1}, 6] -
  2 mm[{1}, 6] mm[{1, 1}, 6] mm[{2, 1}, 6] - mm[{1}, 6]^2 mm[{2, 2}, 6] +
```

```
  mm[{2}, 6] mm[{2, 2}, 6] + mm[{1, 1}, 6] mm[{2, 2}, 6] - mm[{1}, 6]^2 mm[{3, 1}, 6] +
  mm[{2}, 6] mm[{3, 1}, 6] + mm[{1, 1}, 6] mm[{3, 1}, 6] + 2 mm[{1}, 6]^3 mm[{1, 1, 1}, 6] -
  2 mm[{1}, 6] mm[{2}, 6] mm[{1, 1, 1}, 6] - 2 mm[{1}, 6] mm[{1, 1}, 6] mm[{1, 1, 1}, 6] -
  mm[{1}, 6]^2 mm[{2, 1, 1}, 6] + mm[{2}, 6] mm[{2, 1, 1}, 6] +
  mm[{1, 1}, 6] mm[{2, 1, 1}, 6] - mm[{1}, 6]^2 mm[{1, 1, 1, 1}, 6] +
  mm[{2}, 6] mm[{1, 1, 1, 1}, 6] + mm[{1, 1}, 6] mm[{1, 1, 1, 1}, 6],
 mm[{1}, 6]^6 - 3 mm[{1}, 6]^4 mm[{2}, 6] + mm[{1}, 6]^2 mm[{2}, 6]^2 +
  2 mm[{1}, 6]^3 mm[{3}, 6] - mm[{1}, 6]^2 mm[{4}, 6] - 3 mm[{1}, 6]^4 mm[{1, 1}, 6] +
  2 mm[{1}, 6]^2 mm[{2}, 6] mm[{1, 1}, 6] + mm[{1}, 6]^2 mm[{1, 1}, 6]^2 +
  2 mm[{1}, 6]^3 mm[{2, 1}, 6] - mm[{1}, 6]^2 mm[{2, 2}, 6] -
  mm[{1}, 6]^2 mm[{3, 1}, 6] + 2 mm[{1}, 6]^3 mm[{1, 1, 1}, 6] -
  mm[{1}, 6]^2 mm[{2, 1, 1}, 6] - mm[{1}, 6]^2 mm[{1, 1, 1, 1}, 6],
 mm[{1}, 6]^6 - 4 mm[{1}, 6]^4 mm[{2}, 6] + 4 mm[{1}, 6]^2 mm[{2}, 6]^2 +
  2 mm[{1}, 6]^3 mm[{3}, 6] - 4 mm[{1}, 6] mm[{2}, 6] mm[{3}, 6] + mm[{3}, 6]^2 -
  4 mm[{1}, 6]^4 mm[{1, 1}, 6] + 8 mm[{1}, 6]^2 mm[{2}, 6] mm[{1, 1}, 6] -
  4 mm[{1}, 6] mm[{3}, 6] mm[{1, 1}, 6] + 4 mm[{1}, 6]^2 mm[{1, 1}, 6]^2 +
  2 mm[{1}, 6]^3 mm[{2, 1}, 6] - 4 mm[{1}, 6] mm[{2}, 6] mm[{2, 1}, 6] +
  2 mm[{3}, 6] mm[{2, 1}, 6] - 4 mm[{1}, 6] mm[{1, 1}, 6] mm[{2, 1}, 6] + mm[{2, 1}, 6]^2 +
  2 mm[{1}, 6]^3 mm[{1, 1, 1}, 6] - 4 mm[{1}, 6] mm[{2}, 6] mm[{1, 1, 1}, 6] +
  2 mm[{3}, 6] mm[{1, 1, 1}, 6] - 4 mm[{1}, 6] mm[{1, 1}, 6] mm[{1, 1, 1}, 6] +
  2 mm[{2, 1}, 6] mm[{1, 1, 1}, 6] + mm[{1, 1, 1}, 6]^2,
 mm[{1}, 6]^6 - 3 mm[{1}, 6]^4 mm[{2}, 6] + 2 mm[{1}, 6]^2 mm[{2}, 6]^2 +
  mm[{1}, 6]^3 mm[{3}, 6] - mm[{1}, 6] mm[{2}, 6] mm[{3}, 6] -
  3 mm[{1}, 6]^4 mm[{1, 1}, 6] + 4 mm[{1}, 6]^2 mm[{2}, 6] mm[{1, 1}, 6] -
  mm[{1}, 6] mm[{3}, 6] mm[{1, 1}, 6] + 2 mm[{1}, 6]^2 mm[{1, 1}, 6]^2 +
  mm[{1}, 6]^3 mm[{2, 1}, 6] - mm[{1}, 6] mm[{2}, 6] mm[{2, 1}, 6] -
  mm[{1}, 6] mm[{1, 1}, 6] mm[{2, 1}, 6] + mm[{1}, 6]^3 mm[{1, 1, 1}, 6] -
  mm[{1}, 6] mm[{2}, 6] mm[{1, 1, 1}, 6] - mm[{1}, 6] mm[{1, 1}, 6] mm[{1, 1, 1}, 6],
 mm[{1}, 6]^6 - 2 mm[{1}, 6]^4 mm[{2}, 6] + mm[{1}, 6]^3 mm[{3}, 6] -
  2 mm[{1}, 6]^4 mm[{1, 1}, 6] + mm[{1}, 6]^3 mm[{2, 1}, 6] + mm[{1}, 6]^3 mm[{1, 1, 1}, 6],
 mm[{1}, 6]^6 - 3 mm[{1}, 6]^4 mm[{2}, 6] + 3 mm[{1}, 6]^2 mm[{2}, 6]^2 - mm[{2}, 6]^3 -
  3 mm[{1}, 6]^4 mm[{1, 1}, 6] + 6 mm[{1}, 6]^2 mm[{2}, 6] mm[{1, 1}, 6] -
  3 mm[{2}, 6]^2 mm[{1, 1}, 6] + 3 mm[{1}, 6]^2 mm[{1, 1}, 6]^2 -
  3 mm[{2}, 6] mm[{1, 1}, 6]^2 - mm[{1, 1}, 6]^3,
 mm[{1}, 6]^6 - 2 mm[{1}, 6]^4 mm[{2}, 6] + mm[{1}, 6]^2 mm[{2}, 6]^2 - 2 mm[{1}, 6]^4 mm[{1, 1}, 6] +
  2 mm[{1}, 6]^2 mm[{2}, 6] mm[{1, 1}, 6] + mm[{1}, 6]^2 mm[{1, 1}, 6]^2,
 mm[{1}, 6]^6 - mm[{1}, 6]^4 mm[{2}, 6] - mm[{1}, 6]^4 mm[{1, 1}, 6], mm[{1}, 6]^6}
```

```
monomProd2Sum[%]
```

```
{mm[{1, 1, 1, 1, 1, 1}, 6], mm[{2, 1, 1, 1, 1}, 6] + 6 mm[{1, 1, 1, 1, 1, 1}, 6],
 mm[{2, 2, 1, 1}, 6] + 4 mm[{2, 1, 1, 1, 1}, 6] + 15 mm[{1, 1, 1, 1, 1, 1}, 6],
 2 mm[{2, 2, 1, 1}, 6] + mm[{3, 1, 1, 1}, 6] + 9 mm[{2, 1, 1, 1, 1}, 6] +
  30 mm[{1, 1, 1, 1, 1, 1}, 6], mm[{2, 2, 2}, 6] + 2 mm[{2, 2, 1, 1}, 6] +
  6 mm[{2, 1, 1, 1, 1}, 6] + 20 mm[{1, 1, 1, 1, 1, 1}, 6],
 3 mm[{2, 2, 2}, 6] + mm[{3, 2, 1}, 6] + 8 mm[{2, 2, 1, 1}, 6] +
  3 mm[{3, 1, 1, 1}, 6] + 22 mm[{2, 1, 1, 1, 1}, 6] + 60 mm[{1, 1, 1, 1, 1, 1}, 6],
 6 mm[{2, 2, 2}, 6] + 3 mm[{3, 2, 1}, 6] + mm[{4, 1, 1}, 6] + 18 mm[{2, 2, 1, 1}, 6] +
  10 mm[{3, 1, 1, 1}, 6] + 48 mm[{2, 1, 1, 1, 1}, 6] + 120 mm[{1, 1, 1, 1, 1, 1}, 6],
 mm[{3, 3}, 6] + 6 mm[{2, 2, 2}, 6] + 3 mm[{3, 2, 1}, 6] + 15 mm[{2, 2, 1, 1}, 6] +
  6 mm[{3, 1, 1, 1}, 6] + 36 mm[{2, 1, 1, 1, 1}, 6] + 90 mm[{1, 1, 1, 1, 1, 1}, 6],
 2 mm[{3, 3}, 6] + mm[{4, 2}, 6] + 15 mm[{2, 2, 2}, 6] + 8 mm[{3, 2, 1}, 6] +
  2 mm[{4, 1, 1}, 6] + 34 mm[{2, 2, 1, 1}, 6] + 18 mm[{3, 1, 1, 1}, 6] +
  78 mm[{2, 1, 1, 1, 1}, 6] + 180 mm[{1, 1, 1, 1, 1, 1}, 6],
 6 mm[{3, 3}, 6] + 4 mm[{4, 2}, 6] + mm[{5, 1}, 6] + 36 mm[{2, 2, 2}, 6] +
  22 mm[{3, 2, 1}, 6] + 9 mm[{4, 1, 1}, 6] + 78 mm[{2, 2, 1, 1}, 6] +
  48 mm[{3, 1, 1, 1}, 6] + 168 mm[{2, 1, 1, 1, 1}, 6] + 360 mm[{1, 1, 1, 1, 1, 1}, 6],
 mm[{6}, 6] + 20 mm[{3, 3}, 6] + 15 mm[{4, 2}, 6] + 6 mm[{5, 1}, 6] + 90 mm[{2, 2, 2}, 6] +
  60 mm[{3, 2, 1}, 6] + 30 mm[{4, 1, 1}, 6] + 180 mm[{2, 2, 1, 1}, 6] +
  120 mm[{3, 1, 1, 1}, 6] + 360 mm[{2, 1, 1, 1, 1}, 6] + 720 mm[{1, 1, 1, 1, 1, 1}, 6]}
```

```
(ee[#, 6] & /@ Partitions[6]) == (Transpose[ok].ko) . (mm[#, 6] & /@ Partitions[6]);
% /. ee → e /. mm → m // Expand
```

```
True
```

```
(mm[#, 6] & /@ Partitions[6]) ==
  Inverse[Transpose[ok].ko] . (ee[#, 6] & /@ Partitions[6]);
% /. ee → e /. mm → m // Expand
```

```
True
```

```
ee[{1, 1, 1, 1}, 4] /. ee → e2m
```

```
mm[{4}, 4] + 6 mm[{2, 2}, 4] + 4 mm[{3, 1}, 4] + 12 mm[{2, 1, 1}, 4] + 24 mm[{1, 1, 1, 1}, 4]
```

```
ee[{1, 1, 1, 1}, 4] == % /. mm → m /. ee → e // Expand
```

```
True
```

```
mm[{6}, 6] /. mm → m2e
```

```
-6 ee[{6}, 6] + 3 ee[{3, 3}, 6] + 6 ee[{4, 2}, 6] + 6 ee[{5, 1}, 6] -
 2 ee[{2, 2, 2}, 6] - 12 ee[{3, 2, 1}, 6] - 6 ee[{4, 1, 1}, 6] + 9 ee[{2, 2, 1, 1}, 6] +
 6 ee[{3, 1, 1, 1}, 6] - 6 ee[{2, 1, 1, 1, 1}, 6] + ee[{1, 1, 1, 1, 1, 1}, 6]
```

```
mm[{6}, 6] == % /. mm → m /. ee → e // Expand
```

```
True
```