Well Tempering based on the Werckmeister Definition

Addendum:

Application of Weighted Intervals; the Path from Werckmeister to Vallotti-Tartini.

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Abstract : Historical Well Temperaments with quality thirds, and optimised temperaments worked out in the original paper [1], suffer shortcomings on fifths; mainly the fifths on G and D. In this addendum further optimisation of purity of the diatonic C-major is worked out, by improving fifths on diatonic notes. This is achieved by assigning high weights to these fifths in the RMS calculations. The obtained "temperament" shows no significant quality improvement. In a second approach fifths obtain a weight reaching infinity. This "temperament" has equal fifths on all diatonic notes but B. This "temperament" happens to be the Vallotti-Tartini temperament (1750). The results make evident that Vallotti-Tartini is the single best well temperament, if characteristics of the fifths should be at best, in combination with acceptable major thirds.

Keywords : temperament; extended; just; intonation; musical; Pythagorean; natural harmonic; equal temperament; well temperament; optimal; model; objective; root mean square; RMS; interval; pure; third; fifth; diatonic; tune; wohltemperiert; Werckmeister; Vallotti; Tartini; weight

1 Assignment of Heavy Weights to Fifths on Diatonic Notes

The mean PBP (percentage beating pitch of an interval) of fifths and thirds, if worked out on all twelve notes of an octave, equals the values displayed in table 1, below:

	Fifths	Major Thirds	Minor Thirds
Mean PBP	-0,338585	3,968420	-5,396442

Table 1 ; Mean PBP of Fifths, and Thirds

In the approach below, intervals are assigned with weights that are inversely proportional to the mean impurity in table 1, so that overall weighted impurities of fifths and thirds become equal. These weights are displayed in table 2, below, and the high weight for fifths is striking:

	Fifths	Major thirds	Minor thirds
Weight	15,94... = 5,396... / 0,339...	1,40.. = 5,396... / 3,968...	1 = 5,396... / 5,396...

Table 2 ; Weights assigned to Diatonic Intervals of C-major

The weights of table 2 are used further, for the optimisation of the overall impurity of the diatonic C-major, by elaborating the minimum of the weighted root mean square of the thirteen relevant impurities (as was done in paragraph 2.3 of the original paper [[1]]): these intervals are the fifths on all diatonic notes but B, the major thirds on F, C, G, the minor thirds on A, E, B, D; according the formula for the weighted RMS below.

$$\text{"}\boldsymbol{Weighted\_RMS\_of\_Impurities}\text{"} = \sqrt{\frac{\sum \boldsymbol{weight} \times (\text{"}\boldsymbol{Impurity}\text{"})^2}{\sum \boldsymbol{weight}}}$$

The obtained “temperament” is displayed in table 3 below:

Weighted temperament	C	C#	D	Eb	E	F	F#	G	G#	A	Bb	B
pitch	263,1	277,5	294,4	312,2	329,3	351,2	370,0	393,9	416,3	440,0	468,3	493,4

Table 3

Comparison with historical temperaments, using the recognition procedure described in the original paper ([[1]], paragraph 4.1), reveals that this “temperament” is very close to Kirnberger III, Kirnberger III unequal, Vallotti-Tartini, ... temperaments on top of the list in table 11 of the original paper [[1]], The shortcomings on fifths remain, mainly on G and D. It has become clear that use of the herewith proposed weights has no relevant or sufficient influence on the quality of fifths.

2 Assignment of Weights of Fifths reaching Infinity

The heavier the weights of fifths of the diatonic C-major, the more the impurities of those fifths will approach equality. In the limit, with infinite weights, the impurities become equal, and therefore, so do also the values of the ratios for the concerned fifths. By calling " x " the value of the ratio of the six diatonic fifths, while taking into account that all remaining fifths have a ratio 3/2 [2], the following equation can be stated:

$$6 \log_2(3/2) + 6 \log_2(x) = 7$$

The above equation means nothing more but the mathematical expression that twelve superimposed fifths should equal seven octaves, exactly.

Solution of the equation is:

$$x = 2^{(13/6)} / 3 = 1.496616064$$

The obtained "temperament" is displayed in table 4 below.

Weighted temperament	C	C#	D	Eb	E	F	F#	G	G#	A	Bb	B
pitch	262.5	277.2	294.0	311,8	329,3	350,8	369,6	392,9	415,8	440.0	467,7	492,8

Table 4

This "temperament" is IDENTICAL with the Vallotti-Tartini temperament.

Vallotti-Tartini (1750) therefore appears to be the well temperament with optimal combination of purities of thirds and fifths for the diatonic C-major, if the upper best quality of fifths on diatonic notes of C-major is desired.

The remarkable characteristics of intervals of Vallotti-Tartini are displayed in figure 1 below.

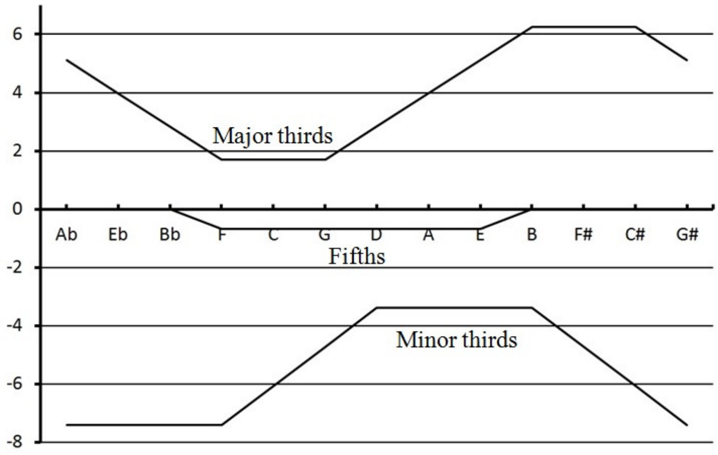

Fig. 1 : Vallotti-Tartini / PBP of intervals

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1 Broekaert Johan: “Well Tempering based on the Werckmeister Definition”,
http://users.telenet.be/broekaert-devriendt/Well_temperament_Werckmeister.pdf
http://home.deds.nl/~broekaert/Well_temperament_Werckmeister.pdf
https://www.academia.edu/6732413/Well_Tempering_based_on_the_Werckmeister_Definition

2 This ratio is = 1.5, see the original paper [1], paragraph 2.3.2: “This specific characteristic, (*the characteristic that those six fifths maintain a pure ratio = 1.5*), will furthermore persist in the following developments of models”

Postamble Tuning Vallotti-Tartini

1 Exact Tuning of Vallotti-Tartini (with A = 440)

The beating of all diatonic fifths or fourths can easily be determined because of the characteristic these all have equal PBP, hence also equal deviation if measured in cents. Values of the beating pitches are given in table 1 below:

	C	C#	D	Eb	E	F	F#	G	G#	A	Bb	B
Fifths	-1.8	0	-2.0	0	-2.2	-2.4	0	-2.7	0	-3.0	0	0
Fourths	2.4	0	2.7	0	3.0	0	0	3.6	0	4.0	0	4.5

Table 1, Beating Pitches Vallotti-Tartini

Possible tuning sequence: (all notes within the octave with A = 440.0)

Tuned Note	Tuning Interval	Beat	Remark
D	Fifth D-A	-2	D slightly high
G	Fourth D-G	2.7	G slightly high
C	Fifth C-G	-1.8	C slightly high
F	Fourth C-F	2.4	F slightly high
Bes	Fourth F-Bes	0	
Es	Fifth Es-Bes	0	
As (Gis)	Fourth Es-As (Gis)	0	
E	Fourth E-A	3.0	E slightly low
B	Fifth E-B	- 2.2	B slightly low
Fis	Fourth Fis-B	0	
Cis	Fourth Cis-Fis	0	
[*Gis (As)*]	[*Fifth Cis-Gis (As)*]	0 (*or very low*)	***Control point***

2 Approximate Tuning of Vallotti Tartini

The exact tuning scheme of Vallotti-Tartini requires some effort to memorise, because of ***differing beat PITCHES***, despite the equal PBP's or cent deviations.

One might wonder about the result if all beating pitches would be set equal.

Equal beat pitch on all diatonic fifths can be obtained by solving following equations, in wich " Beat " is the equal beat pitch:

- The Six wanted pure fifths are obtained by stating:

  4Fis - 3B = 4Cis - 3Fis = 2Gis - 3Cis = 4Es - 3Gis = 2 Bes - 3Es = 4F - 3Bes = 0

  One outcome of this first set of equations, is the fixed relation between F and B:

  $F = (3^6/ 2^{10})\ B$

- For the beating diatonic fifths we can state following equations

  Beat + 4C - 3F = 0; replacing F by $(3^6/ 2^{10})$ B, this becomes:

  Beat - $(3^6/ 2^{10})$ B + 4C = 0, and further on, for the remaining diatonic fifths:

  Beat - 3C + 2G = 0

  Beat +4D - 3G = 0

  Beat - 3D = - 2A = -880

  Beat + 4E = 3A = 1320

  Beat + 2B - 3E = 0

Solution of the above set of equations is given by the formula:

$$\text{Beat} = 440\ (2^{19} - 3^{12})\ /\ (7.3^{10} + 3^3.2^{13} + 23.2^{15}) = -2{,}267209627$$

The obtained "temperament" is displayed in table 2 below:

	C	C#	D	Eb	E	F	F#	G	G#	A	Bb	B
pitch	262.7	277.3	294.1	312.0	329.4	351.0	369.8	392.9	416.0	440.0	468.0	493.0

Table 2 ; "Equally Beating" "Vallotti-Tartini"

Comparison with historical temperaments shows the "temperament" of table 2 is very close to the Vallotti-Tartini temperament. No other historical temperament comes closer to Vallotti-Tartini than the one defined above in table 2.

Advantage of this "temperament", is that it makes tuning more easy: tune the diatonic fourths and fifths in the right sequence, with a beat pitch of approximately 2.27, and finalise the tuning with pure fourths and fifths on the remaining notes, using the latest fifth as control point, where no beating is allowed neither (in reality: a very low beat pitch is required, as evidence that tuning was sufficiently precise).

Possible tuning sequence: (all notes within the octave with A = 440.0)

Tuned Note	Tuning Interval	Beat	Remark
D	Fifth D-A	-2.27	D slightly high
G	Fourth D-G	2.27	G slightly high
C	Fifth C-G	-2.27	C slightly high
F	Fourth C-F	2.27	F slightly high
Bes	Fourth F-Bes	0	
Es	Fifth Es-Bes	0	
As (Gis)	Fourth Es-As (Gis)	0	
E	Fourth E-A	2.27	E slightly low
B	Fifth E-B	- 2.27	B slightly low
Fis	Fourth Fis-B	0	
Cis	Fourth Cis-Fis	0	
[*Gis (As)*]	[*Fifth Cis-Gis (As)*]	0 (*or very low*)	***Control point***

Note: a beat of 2,27 corresponds with 2 beats for one measure, with the metronome set at 53 measures per minute (largo).

3 Musical Evaluation

Musical criticism on Vallotti-Tartini is widely available. Specific, because in line with this paper, but by using musical and historical arguments instead of mathematics, is Carey Beebe, 2013, recommending piano tuners to test Vallotti-Tartini in general, instead of Equal Temperament. See: http://www.hpschd.nu/pdf/tech/vallotti.pdf

"Vallotti's Temperament: Something painless for modern piano technicians to tune"