Convolution SOR waveform relaxation on spatial finite element meshes

This paper investigates the convergence properties of the convolution SOR waveform relaxation method, applied to a system of ordinary differential equations, obtained by spatial finite element discretisation of a linear parabolic initial boundary value problem. We consider both the continuous-time and discrete-time cases and provide a model problem analysis for the one-dimensional heat equation.

1. Introduction

We consider the numerical solution of a linear parabolic initial boundary value problem, spatially discretised by a conforming Galerkin finite element method. This leads to a linear system of ordinary differential equations (ODEs)

\[ B \dot{u}(t) + Au(t) = f(t) , \quad u(0) = u_0 , \]

with \( B = (b_{ij})_{i,j=1}^{d} \) the symmetric positive definite mass matrix and \( A = (a_{ij})_{i,j=1}^{d} \) the stiffness matrix, [7]. For such systems of ODEs, the standard waveform relaxation method and its multigrid acceleration are investigated in [2, 3]. In this paper, we study the convolution SOR (CSOR) waveform relaxation method, introduced in [6] by Reichelt et al., for general ODE-systems of the form (1). This paper is a summary of [4], where a more detailed study of the method, including proofs and a more extensive reference list, can be found. We begin in §2 by describing the continuous-time and discrete-time CSOR waveform relaxation algorithms. The convergence analysis of these methods is outlined in §3. The paper ends in §4, where theoretical and numerical results are given for the one-dimensional heat equation.

2. CSOR waveform relaxation algorithms

The continuous-time CSOR waveform relaxation algorithm for system (1) computes the new waveform approximation \( u^{(v)}_i(t) \), \( 1 \leq i \leq d \), from the previous approximation in two steps. The first step consists of a Gauss-Seidel like computation of \( \hat{u}^{(v)}_i(t) \) along a continuous time-interval,

\[ \left( b_{ii} \frac{d}{dt} + a_{ii} \right) \hat{u}^{(v)}_i(t) = - \sum_{j=1}^{i-1} \left( b_{ij} \frac{d}{dt} + a_{ij} \right) u^{(v)}_j(t) - \sum_{j=i+1}^{d} \left( b_{ij} \frac{d}{dt} + a_{ij} \right) u^{(v-1)}_j(t) + f_i(t) . \]

In the second step the old approximation \( u^{(v-1)}_i(t) \) is updated. Whereas a standard SOR method would extrapolate the correction by multiplying with an overrelaxation parameter \( \omega \), [4], here we convolute the correction with a time-dependent kernel \( \Omega(t) \),

\[ u^{(v)}_i(t) = u^{(v-1)}_i(t) + \int_0^t \Omega(t - \tau) \cdot \left( \hat{u}^{(v)}_i(\tau) - u^{(v-1)}_i(\tau) \right) d\tau . \]

The discrete-time CSOR waveform relaxation algorithm is obtained, for example, by applying a linear multistep method, [1], to (2). Hence, the first step becomes

\[ \sum_{l=0}^{k} \left( \frac{1}{\tau} \alpha_l b_{il} + \beta \alpha_{il} \right) \hat{u}^{(v)}_i[n + l] = - \sum_{j=1}^{i-1} \sum_{l=0}^{k} \left( \frac{1}{\tau} \alpha_l b_{ij} + \beta \alpha_{ij} \right) u^{(v)}_j[n + l] \]
\[ - \sum_{j=i+1}^{d} \sum_{l=0}^{k} \left( \frac{1}{\tau} \alpha_l b_{ij} + \beta \alpha_{ij} \right) u^{(v-1)}_j[n + l] + \sum_{l=0}^{k} \beta_l f_i[n + l] , \]

where \( \alpha_l \) and \( \beta_l \) are the coefficients of the multistep method, \( \beta \) is the (constant) step-size, and \( u^{(v)}_i[n] \) denotes the discrete approximation of \( u^{(v)}_i(t) \) at \( t = n\tau \). In the second step, the continuous-time convolution is replaced by its
3. Convergence analysis

By rewriting the continuous-time iterative scheme (2)-(3) in explicit form, we can derive that the CSOR iteration operator $K_{CSOR}$ consists of a matrix multiplication and a linear Volterra convolution part,

$$u^{(v)}(t) = K^{CSOR}u^{(v-1)}(t) + \mathcal{V}(u^{(v-1)}(t)) + \varphi(t),$$

where $\mathcal{V}$ consists of a matrix multiplication and a linear Volterra convolution part, $K^{CSOR}$ is the continuous-time CSOR symbol equals

$$K^{CSOR}(z) = \left( z \left( \frac{1}{\Omega(z)} D_B - L_B \right) + \left( \frac{1}{\Omega(z)} D_A - L_A \right) \right)^{-1} \,,$$

with $B = D_B - L_B - U_B$ and $A = D_A - L_A - U_A$ the standard splittings of $B$ and $A$ in diagonal, lower and upper triangular parts. In terms of this symbol, we can prove the following convergence theorem on infinite time-intervals.

**Theorem 1.** Let $K^{CSOR}$ be an operator in $L_p[0,\infty)$, $1 \leq p \leq \infty$. Then, $K^{CSOR}$ is a bounded operator and

$$\rho(K^{CSOR}) = \sup_{Re(z) \geq 0} \rho(K^{CSOR}(z)) = \sup_{\xi \in \mathbb{R}} \rho(K^{CSOR}(i\xi)) \,.$$  

The following expression for the Laplace-transform of the optimal convolution kernel $\Omega_{opt}(t)$ can be derived, [4].

**Theorem 2.** Assume $B$ and $A$ are such that $zB + A$ is consistently ordered, $\det(zD_B + D_A) \neq 0$, and the spectrum $\sigma(K^{JAC}(z)) = \sigma((zD_B + D_A)^{-1}((zL_B + U_B) + (L_A + U_A)))$ lies on a line segment $[-\mu_0(z), \mu_0(z)]$ with $\mu_0(z) \in \mathbb{C}$ and $|\mu_0(z)| < 1$. The spectral radius of $K^{CSOR}(z)$ is then minimised by the unique optimum $\hat{\Omega}_{opt}(z)$, and is given by

$$\rho(K^{CSOR},\hat{\Omega}_{opt}(z)) = |\hat{\Omega}_{opt}(z) - 1| < 1 \,,$$

where $\sqrt{\cdot}$ denotes the root with the positive real part.

A similar analysis can be done for the discrete-time CSOR waveform relaxation method. The operator $K^{CSOR}$ turns out to be of discrete convolution type, as we can rewrite (4)-(5) into the following form

$$u^{(v)}[n] = K^{CSOR}[n] + \varphi[n] = \sum_{i=0}^{n} \left( k[n - i] u^{(v-1)}[i] + \varphi[n] \right).$$

The discrete-time CSOR symbol $K^{CSOR}(z)$ is obtained by discrete Laplace- or Z-transformation of (9). More precisely, we have $\tilde{u}^{(v)}(z) = K^{CSOR}(z)\tilde{u}^{(v-1)}(z) + \tilde{\varphi}(z)$, where $\tilde{u}^{(v)}(z)$ denotes the Z-transform of the sequence $u^{(v)} = \{u^{(v)}[0], u^{(v)}[1], u^{(v)}[2], \ldots\}$ and

$$K^{CSOR}_d(z) = \left( \frac{a(z)}{b(z)} \left( \frac{1}{\Omega(z)} D_B - L_B \right) + \left( \frac{1}{\Omega(z)} D_A - L_A \right) \right)^{-1} \,,$$

with $a(z) = \sum_{j=0}^{k} \alpha_j z^j$ and $b(z) = \sum_{j=0}^{k} \beta_j z^j$ the characteristic polynomials of the multistep method. We then obtain the following discrete-time equivalents of Theorems 1 and 2.
Theorem 3. Let $K_{C}\text{SOR}$ be an operator in $l_p(\infty)$, $1 \leq p \leq \infty$. Then $K_{C}\text{SOR}$ is a bounded operator and

$$\rho(K_{C}\text{SOR}) = \max_{|z|=1} \rho(K_{C}\text{SOR}(z)) = \max_{|z|=1} \rho(K_{C}\text{SOR}(z)) .$$

Theorem 4. Assume $B$ and $A$ are such that $\frac{1}{\tau} B(z) D_B + D_A \neq 0$, and the spectrum $\sigma(K_{B}\text{AC}(\frac{1}{\tau} B(z))) = \sigma(K_{B}\text{AC}(\frac{1}{\tau} B(z)))$ lies on a line segment $[-(\mu_1)_{\tau}(z), (\mu_1)_{\tau}(z)]$ with $(\mu_1)_{\tau}(z) \in \mathbb{C}$ and $|(\mu_1)_{\tau}(z)| < 1$. The spectral radius of $K_{C}\text{SOR}(z)$ is then minimised by the unique optimum $(\hat{\Omega}_{opt})_{\tau}(z)$, and is given by

$$\rho(K_{C}\text{SOR},(\hat{\Omega}_{opt})_{\tau}(z)) = |(\hat{\Omega}_{opt})_{\tau}(z) - 1| < 1 , \quad \text{with} \quad (\hat{\Omega}_{opt})_{\tau}(z) = \frac{2}{1 + \sqrt{1 - (\mu)_{\tau}(z)}},$$

where $\sqrt{\cdot}$ denotes the root with the positive real part.

By comparison of (8) and (11), we observe that $(\hat{\Omega}_{opt})_{\tau}(z) = \hat{\Omega}_{opt}(\frac{1}{\tau} B(z))$. Consequently, in the optimal case, (10) can be rewritten as

$$\rho(K_{C}\text{SOR},(\hat{\Omega}_{opt})_{\tau}) = \sup \left\{ \rho(K_{C}\text{SOR},(\hat{\Omega}_{opt})_{\tau}(z)) \mid \tau z \in \mathbb{C} \setminus \text{intS} \right\} = \sup_{z \in \mathbb{C} \setminus \text{intS}} \rho(K_{C}\text{SOR},(\hat{\Omega}_{opt})_{\tau}(z)) ,$$

where $S$ denotes the stability region of the multistep method.

4. Model problem analysis

Consider the one-dimensional heat equation

$$\frac{\partial u}{\partial t} - \Delta u = 0 , \quad x \in [0,1] , \quad t \in [0,1] ,$$

with homogeneous Dirichlet boundary conditions and a given initial condition. Let the equation be discretised using linear finite element basis functions on a grid with mesh-size $h$, i.e., $\Omega_h = \{x_i = ih \mid 0 \leq i \leq 1/h\}$. This leads to a system of ODEs (1) with (in stencil notation) $B = \left[ \begin{smallmatrix} a & b \\ b & a \end{smallmatrix} \right]$ and $A = \left[ \begin{smallmatrix} 1-rac{1}{h} & \frac{1}{h} \\ \frac{1}{h} & 1-rac{1}{h} \end{smallmatrix} \right]$.

Theorem 5. Consider $K_{C}\text{SOR},(\hat{\Omega}_{opt}(\tau))$ as an operator in $l_p(\infty)$, $1 \leq p \leq \infty$, for solving (13), discretised using linear finite element basis functions. Then, for small $h$, we have that

$$\rho(K_{C}\text{SOR},(\hat{\Omega}_{opt}(\tau))) \approx 1 - 2\pi h .$$

The proof of Theorem 5 is based on the observation that the maximum of $\rho(K_{C}\text{SOR},(\hat{\Omega}_{opt}(\tau)))$ along the imaginary axis is attained at the origin. Hence, $\rho(K_{C}\text{SOR},(\hat{\Omega}_{opt}(\tau)))$ equals the spectral radius of the algebraic SOR method for matrix $A$ with optimal overrelaxation parameter $\hat{\Omega}_{opt}(0)$, which is well-known to be $1 - 2\pi h$ for small $h$, [8].

Table 1 presents some averaged convergence factors obtained with an implementation of the discrete-time convolution SOR waveform relaxation method with optimal convolution kernel, for model problem (13). We used the Crank-Nicolson (CN) method and the backward differentiation (BDF) formulae of order 1, 3 and 5, with time-step $\tau = 1/100$. It is well-known for waveform relaxation methods that the numerical results, although obtained on finite time-intervals, match the infinite time-interval theoretical analysis, see e.g. [5] for a theoretical explanation based on the pseudospectra of the relevant operators. Indeed, the observed convergence factors in Table 1 are in close correspondence with the theoretical result (14). Also, the results of Table 1 show that, for a fixed value of $h$, the observed convergence factors are more or less independent of the chosen time-discretisation method. An explanation of this behaviour can be found in Figure 1, where we visualise the application of formula (12) for model problem (13) by means of a so-called spectral picture: we display contour lines of $\rho(K_{C}\text{SOR},(\hat{\Omega}_{opt}(\tau)))$ for values

<table>
<thead>
<tr>
<th>$h$</th>
<th>1/8</th>
<th>1/16</th>
<th>1/32</th>
<th>1/64</th>
</tr>
</thead>
<tbody>
<tr>
<td>CN</td>
<td>0.386</td>
<td>0.637</td>
<td>0.802</td>
<td>0.899</td>
</tr>
<tr>
<td>BDF (1)</td>
<td>0.389</td>
<td>0.637</td>
<td>0.802</td>
<td>0.896</td>
</tr>
<tr>
<td>BDF (3)</td>
<td>0.378</td>
<td>0.630</td>
<td>0.797</td>
<td>0.884</td>
</tr>
<tr>
<td>BDF (5)</td>
<td>0.317</td>
<td>0.620</td>
<td>0.797</td>
<td>0.894</td>
</tr>
<tr>
<td>$1 - 2\pi h$</td>
<td>0.215</td>
<td>0.607</td>
<td>0.804</td>
<td>0.902</td>
</tr>
</tbody>
</table>

Table 1: Observed convergence factors for (13) - optimal CSOR waveform relaxation - $\tau = 1/100$. 

0.6, 0.7, 0.8 and 0.9), together with (parts of) the scaled stability region boundaries \( \frac{1}{\tau} \partial S \) of the CN method and the BDF methods of order 1, 3 and 5.

![Figure 1: Spectral picture for (13) - optimal CSOR waveform relaxation - \( h = 1/16, \tau = 1/100 \).](image)

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5. **References**


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