10.8 Relativity without c

In Section 10.1, we introduced the two postulates of Special Relativity, namely the speed-of-light postulate and the relativity postulate. In Appendix I, we show that together these imply that the coordinate intervals in two frames must be related by the Lorentz transformations, eqs. (10.13).

It is interesting to see what happens if we relax these postulates. It is hard to imagine a "reasonable" universe where the relativity postulate does not hold (geocentric theories aside), but it is easy to imagine a universe where the speed of light depends on the frame of reference. Light could behave like a baseball, for example. So let's drop the speed-of-light postulate and see what we can say about the coordinate transformations between frames, using only the relativity postulate.

In Appendix I, the form of the transformations, just prior to invoking the speed-of-light postulate, is given in eq. (14.83) as

$$\begin{aligned} x &= A_v(x' + vt'), \\ t &= A_v\left(t' + \frac{1}{v}\Big(1 - \frac{1}{A_v^2}\Big)x'\right). \end{aligned} \qquad (10.66)$$

We'll put a subscript on A in this section, to remind us of the v dependence. Can we say anything about A_v without invoking the speed-of-light postulate? Indeed we can. Define V_v by

$$\frac{1}{V_v^2} \equiv \frac{1}{v^2}\left(1 - \frac{1}{A_v^2}\right), \qquad \text{so that} \qquad A_v = \frac{1}{\sqrt{1 - v^2/V_v^2}}. \tag{10.67}$$

Eqs. (10.66) then become

$$\begin{aligned} x &= \frac{1}{\sqrt{1 - v^2/V_v^2}}(x' + vt'), \\ t &= \frac{1}{\sqrt{1 - v^2/V_v^2}}\left(\frac{v}{V_v^2}x' + t'\right). \end{aligned} \tag{10.68}$$

All we've done so far is make a change of variables. But we now make the following claim.

Claim 10.1 *V_v^2 is independent of v.*

Proof: As stated in the last remark in Section 10.3.1, we know that two successive applications of the transformations in eq. (10.68) must again yield a transformation of the same form.

Consider a transformation characterized by velocity v_1, and another one characterized by velocity v_2. For simplicity, define

$$\begin{aligned} V_1 &\equiv V_{v_1}, & V_2 &\equiv V_{v_2}, \\ \gamma_1 &\equiv \frac{1}{\sqrt{1 - v_1^2/V_1^2}}, & \gamma_2 &\equiv \frac{1}{\sqrt{1 - v_2^2/V_2^2}}. \end{aligned} \tag{10.69}$$

To calculate the composite transformation, it is easiest to use matrix notation. Looking at eq. (10.68), we see that the composite transformation is given by the matrix

$$\begin{pmatrix} \gamma_2 & \gamma_2 v_2 \\ \gamma_2\frac{v_2}{V_2^2} & \gamma_2 \end{pmatrix} \begin{pmatrix} \gamma_1 & \gamma_1 v_1 \\ \gamma_1\frac{v_1}{V_1^2} & \gamma_1 \end{pmatrix} = \gamma_1\gamma_2 \begin{pmatrix} 1 + \frac{v_1 v_2}{V_1^2} & v_2 + v_2 \\ \frac{v_1}{V_1^2} + \frac{v_2}{V_2^2} & 1 + \frac{v_1 v_2}{V_2^2} \end{pmatrix}. \tag{10.70}$$

The composite transformation must still be of the form of eq. (10.68). But this implies that the upper-left and lower-right entries of the composite matrix must be equal. Therefore, $V_1^2 = V_2^2$. Since this holds for arbitrary v_1 and v_2, we see that V_v^2 must be a constant, that is, independent of v. ■

Denote the constant value of V_v^2 by V^2. Then the coordinate transformations in eq. (10.68) become

$$\begin{aligned} x &= \frac{1}{\sqrt{1 - v^2/V^2}}(x' + vt'), \\ t &= \frac{1}{\sqrt{1 - v^2/V^2}}\left(t' + \frac{v}{V^2}x'\right). \end{aligned} \tag{10.71}$$

We have obtained this result using only the relativity postulate. These transformations have the same form as the Lorentz transformations, eqs. (10.13). The only extra information in eqs. (10.13) is that V is equal to the speed of light, c. It is remarkable that we were able to prove so much by using only the relativity postulate.

We can say a few more things. There are four possibilities for the value of V^2. However, two of these are not physical.

- $V^2 = \infty$: This gives the Galilean transformations, $x = x' + vt'$ and $t = t'$.
- $0 < V^2 < \infty$: This gives transformations of the Lorentz type. V is the limiting speed of an object.
- $V^2 = 0$: This case is not physical, because any nonzero value of v will make the γ factor imaginary (and infinite). Nothing could ever move.
- $V^2 < 0$: It turns out that this case is also not physical. You might be concerned that the square of V is less than zero, but this is fine because V appears in the transformations (10.71) only through its square. There's no need for V to actually be the speed of anything. The trouble is that the nature of eqs. (10.71) implies the possibility of time reversal. This opens the door for causality violation and all the other problems associated with time reversal. We therefore reject this case.

  To be a little more explicit, define $b^2 \equiv -V^2$, where b is a positive number. Then eqs. (10.71) may be written in the form,

$$\begin{aligned} x &= x' \cos\theta + (bt')\sin\theta, \\ bt &= -x' \sin\theta + (bt')\cos\theta, \end{aligned} \qquad (10.72)$$

  where $\tan\theta = v/b$. This transformation is simply a rotation in the plane, through an angle of $-\theta$. We have the usual trig functions here, instead of the hyperbolic trig functions in the Lorentz transformations in eq. (10.55).

  Eqs. (10.72) satisfy the requirement that the composition of two transformations is again a transformation of the same form. Rotation by θ_1, and then by θ_2, yields a rotation by $\theta_1 + \theta_2$. However, if the resulting rotation is through an angle, θ, that is greater than 90°, then we have a problem. The tangent of such an angle is negative. Therefore, $\tan\theta = v/b$ implies that v is negative.

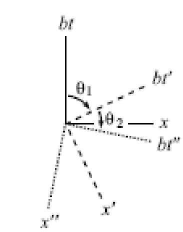

Figure 10.29

  The situation is shown in Fig. 10.29. Frame S'' moves at velocity $v_2 > 0$ with respect to frame S', which moves at velocity $v_1 > 0$ with respect to frame S. From the figure, we see that someone standing at rest in frame S'' (that is, someone whose worldline is the bt''-axis) is going to have some serious issues in frame S. For one, the bt''-axis has a negative slope in frame S, which means that as t increases, x decreases. The person will therefore be moving at a *negative* velocity with respect to S. Adding two positive velocities and obtaining a negative one is clearly absurd. But even worse, someone standing at rest in S'' will be moving in the positive direction along the bt''-axis, which means that he will be traveling *backwards* in time in S. That is, he will die before he is born. This is not good.

Note that all of the finite $0 < V^2 < \infty$ possibilities are essentially the same. Any difference in the numerical definition of V can be absorbed into the definitions of the unit sizes for x and t. Given that V is finite, it has to be *something*, so it doesn't make sense to put much importance on its numerical value.

There is therefore only one decision to be made when constructing the spacetime structure of an (empty) universe. You just have to say whether V is finite or infinite, that is, whether the universe is Lorentzian or Galilean. Equivalently, all you have to say is whether or not there is an upper limit for the speed of any object. If there is, then you can simply postulate the existence of something that moves with this limiting speed. In other words, to create your universe, you simply have to say, "Let there be light."

14.9 Appendix I: Lorentz transformations

In this Appendix, we will give an alternate derivation of the Lorentz transformations, eqs. (10.13). The goal here is to derive them from scratch, using only the two postulates of relativity. We will *not* use any of the results derived in Section 10.2. Our strategy will be to use the relativity postulate ("all inertial frames are equivalent") to figure out as much as we can, and to then invoke the speed-of-light postulate at the end. The main reason for doing things in this order is that it will allow us to derive an very interesting result in Section 10.8.

As in Section 10.3, consider a coordinate system, S', moving relative to another system, S (see Fig. 14.9). Let the constant relative speed of the frames be v. Let the corresponding axes of S and S' point in the same direction, and let the origin of S' move along the x-axis of S, in the positive direction.

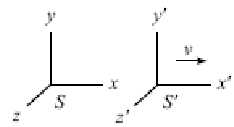

Figure 14.9

As in Section 10.3, we want to find the constants, A, B, C, and D, in the relations,

$$\begin{aligned} \Delta x &= A\,\Delta x' + B\,\Delta t', \\ \Delta t &= C\,\Delta t' + D\,\Delta x'. \end{aligned} \tag{14.76}$$

The four constants here will end up depending on v (which is constant, given the two inertial frames).

There are four unknowns in eqs. (14.76), so we need four facts. The facts we have at our disposal (using only the two postulates of relativity) are the following.

1. The physical setup: S' travels at speed v with respect to S.
2. The principle of relativity: S should see things in S' in exactly the same way as S' sees things in S (except for perhaps a minus sign in some relative positions, but this is just convention).
3. The speed-of-light postulate: A light pulse with speed c in S' also has speed c in S.

The second statement here contains two independent bits of information. (It contains at least two, because we will indeed be able to solve for our four unknowns. And it contains no more than two, because then our four unknowns would be overconstrained.) The two bits that are used depend on personal preference. Three that are commonly used are: (a) the relative speed looks the same from either frame, (b) time dilation (if any) looks the same from either frame, and (c) length contraction (if any) looks the same from either frame. It is also common to recast the second statement in the form: The Lorentz transformations are equal to their inverse transformations (up to a possible minus sign). We'll choose to work with (a) and (b). Our four independent facts are then:

1. S' travels at speed v with respect to S.
2. S travels at velocity $-v$ with respect to S'. The minus sign here is due to the convention that we picked the positive x-axes of the two frames to point in the same direction.

3. Time dilation (if any) looks the same from either frame.

4. A light pulse with speed c in S' also has speed c in S.

Let's see what these imply, in the above order.[14]

- (1) says that a given point in S' moves at speed v with respect to S. Letting $x' = 0$ (which is understood to be $\Delta x' = 0$, but we'll drop the Δ's from here on) in eqs. (14.76) gives $x/t = B/C$. This must equal v. Therefore, $B = vC$, and the transformations become

$$\begin{aligned} x &= Ax' + vCt', \\ t &= Ct' + Dx'. \end{aligned} \tag{14.77}$$

- (2) says that a given point in S moves at velocity $-v$ with respect to S'. Letting $x = 0$ in the first of eqs. (14.77) gives $x'/t' = -vC/A$. This must equal $-v$. Therefore, $C = A$, and the transformations become

$$\begin{aligned} x &= Ax' + vAt', \\ t &= At' + Dx'. \end{aligned} \tag{14.78}$$

- (3) can be used in the following way. How fast does a person in S see a clock in S' tick? (The clock is assumed to be at rest with respect to S'.) Let our two events be two successive ticks of the clock. Then $x' = 0$, and the second of eqs. (14.78) gives

$$t = At'. \tag{14.79}$$

In other words, one second on S''s clock takes a time of A seconds in S's frame.

Consider the analogous situation from S''s point of view. How fast does a person in S' see a clock in S tick? (The clock is now assumed to be at rest with respect to S, in order to create the analogous setup. This is important.) If we invert eqs. (14.78) to solve for x' and t' in terms of x and t, we find

$$\begin{aligned} x' &= \frac{x - vt}{A - vD}, \\ t' &= \frac{At - Dx}{A(A - vD)}. \end{aligned} \tag{14.80}$$

Two successive ticks of the clock in S satisfy $x = 0$, so the second of eqs. (14.80) gives

$$t' = \frac{t}{A - vD}. \tag{14.81}$$

In other words, one second on S's clock takes a time of $1/(A - vD)$ seconds in S''s frame.

[14] In what follows, we could obtain the final result a little quicker if we invoked the speed-of-light fact prior the time-dilation one. But we'll do things in the above order so that we can easily carry over the results of this Appendix to the discussion in Section 10.8.

Both eqs. (14.79) and (14.81) apply to the same situation (someone looking at a clock flying by). Therefore, the factors on the right-hand sides must be equal, that is,

$$A = \frac{1}{A - vD} \qquad \Longrightarrow \qquad D = \frac{1}{v}\left(A - \frac{1}{A}\right). \tag{14.82}$$

Our transformations in eqs. (14.78) therefore take the form

$$\begin{aligned} x &= A(x' + vt'), \\ t &= A\left(t' + \frac{1}{v}\Big(1 - \frac{1}{A^2}\Big)x'\right). \end{aligned} \tag{14.83}$$

- (4) may now be used to say that if $x' = ct'$, then $x = ct$. In other words, if $x' = ct'$, then

$$c = \frac{x}{t} = \frac{A((ct') + vt')}{A\left(t' + \frac{1}{v}\Big(1 - \frac{1}{A^2}\Big)(ct')\right)} = \frac{c + v}{1 + \frac{c}{v}\Big(1 - \frac{1}{A^2}\Big)}. \tag{14.84}$$

  Solving for A gives

$$A = \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{14.85}$$

  We have chosen the positive square root so that the positive x and x' axes point in the same direction.

The constant A is commonly denoted by γ, so we may finally write our Lorentz transformations, eqs. (14.83), in the form,

$$\begin{aligned} x &= \gamma(x' + vt'), \\ t &= \gamma(t' + vx'/c^2), \end{aligned} \tag{14.86}$$

where

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}, \tag{14.87}$$

in agreement with eq. (10.13).